TAKE OUT YOUR LAPTOP / CELL (I)

GO TO HTTPS://EVAL.OLT.UBC.CA/SCIENCE

& FILL OUT EVALUATION FOR MATH 104

POLL

OFFICE HOURS
(ALSO TOMORROW 2→3)

M 1→2:30
F 1, 2, 3

REVIEW SESSION

W 1, 2, 3
TH 1, 2, 3

LAST TIME
↳ TOPOLOGY

TODAY
↳ ∞ SUMS

Q
- WHAT IS A SUM?
- WHAT IS A FINITE SUM?
- IS IT POSSIBLE TO ADD ∞-LY MANY NUMBERS?
- IF IT WAS POSSIBLE, WHAT WOULD YOU EXPECT TO GET?

EX

$$1+\frac{1}{2} = 1+\frac{1}{2} = 1.5$$

$$1+\frac{1}{2}+\frac{1}{4} = 1+\frac{3}{4} = 1.75$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 1+\frac{7}{8} = 1.875$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = 1+\frac{15}{16} = 1.9375$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} = 1+\frac{31}{32} = 1.96875$$

$$1+\frac{1}{2}+\cdots+\frac{1}{32}+\frac{1}{64} = 1+\frac{63}{64} = 1.984375$$

$$1+\frac{1}{2}+\cdots+\frac{1}{2^{10}} = 1.9990234375$$

$$1+\frac{1}{2}+\cdots+\frac{1}{2^{100}} = 1.999999999999\cdots \text{STUFF}$$

PRETTY CLEAR THAT THESE NUMBERS ARE GOING SOMEWHERE

WE WOULD LIKE TO TAKE A LIMIT OR SOMETHING ...

BUT HOW ?

Q WHAT DO WE KNOW HOW TO TAKE LIMITS WITH? III

$\Rightarrow$ FUNCTIONS

Q WHAT FUNCTION SHOULD WE USE?

$$\mathbb{N} \xrightarrow{f} \mathbb{R}$$

$$f(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$$ PARTIAL SUM

= SUM OF FIRST n-TERMS

IN $\underbrace{1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots}_{\sum_{n=1}^{\infty} \frac{1}{2^n}}$

IF $\lim_{n \to \infty} f(n)$ EXISTS WE

SAY $\;\; \| \;\; 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots$

& HAVE COMPUTED THE INFINITE SUM

HARD TO COMPUTE $f(n)$

BUT HERE THERE'S A TRICK

$$\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}=f(n)=?$$

$$\left(1-\frac{1}{2}\right)\left(\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}+\frac{1}{2^n}\right)=\left(1-\frac{1}{2}\right)f(n)$$

$$\frac{1}{2^0}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{4}}+\cancel{\frac{1}{4}}\cdots\cancel{+\frac{1}{2^{n-1}}}-\frac{1}{2^{n+1}}$$

$$\left(1-\frac{1}{2^{n+1}}\right)=\left(1-\frac{1}{2}\right)f(n)$$

$$\Rightarrow \boxed{\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\left(\frac{1}{2}\right)}=f(n)}$$

MAGIC

$$\lim_{n\to\infty}\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-1/2}=\lim_{n\to\infty}\frac{1}{1-1/2}-\lim_{n\to\infty}\frac{\left(1/2\right)^{n+1}}{1-1/2}$$

$$=\frac{1}{1-1/2}-0=\boxed{2}$$

V

ONLY THING WE USED ABOUT $\frac{1}{2}$ WAS THAT $0 < \frac{1}{2} < 1$

$\Rightarrow$ IF $0 < r < 1$

THEN $\underbrace{1 + r + r^2 + r^3 + \cdots + r^n}_{\frac{1-r^{n+1}}{1-r} \xrightarrow[n\to\infty]{} \frac{1}{1-r}} + \cdots = \frac{1}{1-r}$

SAME AS POLY BUT ∞-LY MANY TERMS

$\boxed{\text{THIS IS A TAYLOR SERIES AT } 0 \text{ FOR } f(x) = \frac{1}{1-x} \text{ VALID WHEN } 0<x<1}$

$$P_n(x) = f(0) + f'(0)(x-0) + \frac{f''(0)(x-0)^2}{2} + \cdots + \frac{f^{(n)}(0)(x-0)^n}{n+1}$$

$$f'(x) = \frac{1}{(1-x)^2} \qquad f''(x) = \frac{-2(1-x)(-1)}{(1-x)^4}$$

$$f'(0) = 1 \qquad f''(0) = 2 \text{ \& SO ON}$$

CAUTION: OUTSIDE $0<r<1$

MAYBE THE SUM BLOWS UP

OR DIVERGES

FOR VERY NICE FUNCTIONS DON'T NEED TO WORRY ABOUT DOMAIN

FAVOURITE EXAMPLES: TAYLOR AT ZERO

$$e^x = \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}}_{} + \cdots$$

FORMULA IS VALID EVERYWHERE

$f(x)$ HAS A LIMIT FOR ALL x

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

↳ ODD DEGREE TERMS OF e^x

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

↳ EVEN DEGREE TERMS OF e^x

EVEN VS ODD FUNCTION

WHO KNOWS ABOUT IMAGINARY NUMBERS? $\boxed{i^2 = -1}$ JUST LIKE $\boxed{(\sqrt{2})^2 = 2}$ **VII**

$1 + x^2 = 0$ HAS A SOLUTION?

LETS DO SOMETHING (FUN) CRAZY

$$e^{\boxed{}} = 1 + \boxed{} + \frac{\boxed{}^2}{2!} + \frac{\boxed{}^3}{3!} + \frac{\boxed{}^4}{4!} + \frac{\boxed{}^5}{5!} + \cdots$$

PLUG IN $\boxed{2x}$ OR $\boxed{3x}$ OR .. $\boxed{ix}$

$$e^{(ix)} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \frac{i^5 x^5}{5!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots$$

WHAT JUST HAPPENED

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$\cos(x)$ $\qquad\qquad$ $i\sin(x)$

A GENUINELY WTF MOMENT.

VIII

$$\boxed{e^{ix} = \cos(x) + i\sin(x)}$$

EULER'S FORMULA, ONE OF THE MOST REMARKABLE FORMULAS IN MATHEMATICS

$$\boxed{x = \pi}$$

$$\Rightarrow e^{i\pi} = \cos(\pi) + i\sin(\pi)$$

-1 0

$$e^{i\pi} = -1 \quad \text{OR}$$

$$\boxed{e^{i\pi} + 1 = 0}$$

FUNDAMENTAL RELATIONSHIP BETWEEN FIVE MOST IMPORTANT CONSTANTS 0, 1, π, e & i

THE END