

Midterm Exam I

October 7, 2011

No books. No notes. No calculators. No electronic devices of any kind.

Name _____

Student Number _____

Problem 1. (6 points)For each of the three matrices A , B , C find all values of the indeterminates, such that the matrix is in reduced row echelon form.

$$A = \begin{pmatrix} 0 & 0 & 1 & y & 0 \\ 0 & x & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$x = 0$ (can't have a pivot in Row 2 to the left of the pivot in Row 1.)

$z = 1$ (can't have a row of zeros above a row with a pivot & pivot needs to be = 1 in rref)

$$B = \begin{pmatrix} 1 & 2 & 0 & y & z \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$y = 0$ (all entries in a pivot column other than the pivot have to be = 0.)

$z = 0$ (all entries in a pivot column other than the pivot have to be = 0)

x can be any real number

y can be any real number

$$C = \begin{pmatrix} 0 & 1 & y & 0 & 0 \\ 0 & 0 & 0 & x & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$z = 0$ (all entries in a pivot column other than the pivot have to be = 0)

$x = 1$ (can't have a row of zeros above a row with a pivot & pivot needs to be = 1 in rref)

y can be any real number

in each case, the pivots are circled.

1	2	3	4	5	6	7	total/40

Problem 2. (6 points)

Solve the following system of linear equations.

Give the answer in **parametric vector form**.

$$\begin{array}{l} 2x_1 + 4x_2 + 2x_3 + 2x_4 = 6 \\ x_1 + 2x_2 + x_3 = 3 \\ x_1 + x_2 - x_3 = 3 \end{array}$$

$$\left(\begin{array}{cccc|c} 2 & 4 & 2 & 2 & 6 \\ 1 & 2 & 1 & 0 & 3 \\ 1 & 1 & -1 & 0 & 3 \end{array} \right) \xrightarrow{\cdot \frac{1}{2}} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 1 & 2 & 1 & 0 & 3 \\ 1 & 1 & -1 & 0 & 3 \end{array} \right) \xrightarrow{-I} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 3 \end{array} \right)$$

$$\xrightarrow{-} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & -2 & -1 & 0 \end{array} \right) \xrightarrow{\leftarrow} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array} \right) \xrightarrow{\cdot (-1)} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right) \xrightarrow{\cdot (-1)} \text{now it's in ref}$$

$$\xrightarrow{-\text{III}} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right) \xrightarrow{-\text{III}} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right) \xrightarrow{-2\text{II}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

$$\xrightarrow{\left(\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 3 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)} \text{now it's in rref.}$$

no pivot in augmentation column: it's consistent.

no pivot in x_3 -column: x_3 is free.

$$x_1 = 3 + 3x_3$$

$$x_2 = -2x_3$$

$$x_3 = x_3 \text{ (free)}$$

$$x_4 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3+3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

general solution in parametric vector form.

Check answer:

$$2(3+3x_3) + 4(-2x_3) + 2 \cdot 0 = 6 + 6x_3 - 8x_3 + 2x_3 = 6 \quad \checkmark$$

$$(3+3x_3) + 2(-2x_3) + x_3 = 3 + 3x_3 - 4x_3 + x_3 = 3 \quad \checkmark$$

$$(3+3x_3) + (-2x_3) - x_3 = 3 + 3x_3 - 2x_3 - x_3 = 3 \quad \checkmark$$

Problem 3. (5 points)

Decide whether or not the vector \vec{v} is in $\text{span}(u_1, u_2)$. In other words, decide if there exist x_1 and x_2 , such that $\vec{v} = x_1 \vec{u}_1 + x_2 \vec{u}_2$. Justify your answer.

$$\vec{v} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \quad \text{same as} \quad \begin{pmatrix} 1 & -3 \\ 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

equivalent to the system of equations with augmented coefficient matrix

$$\left(\begin{array}{cc|c} 1 & -3 & 5 \\ 2 & 1 & 3 \\ 2 & 2 & 2 \end{array} \right) \quad \text{We have to decide if the system is consistent.}$$

$$\left(\begin{array}{cc|c} 1 & -3 & 5 \\ 2 & 1 & 3 \\ 2 & 2 & 2 \end{array} \right) \xrightarrow{-2I} \left(\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 7 & -7 \\ 0 & 8 & -8 \end{array} \right) \xrightarrow{\cdot\frac{1}{7}} \left(\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{array} \right) \xrightarrow{-II} \left(\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{cc|c} \textcircled{1} & -3 & 5 \\ 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{matrix} \text{this is in ref (pivots are circled)} \\ \leftarrow \text{no pivot in augmentation column} \rightarrow \text{system consistent.} \end{matrix}$$

Yes, \vec{v} is in $\text{span}(\vec{u}_1, \vec{u}_2)$.

Even though it was not required to do, we can find x_1 and x_2 :

$$\left(\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right) \xrightarrow{+3I} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{matrix} x_1 = 2 \\ x_2 = -1 \end{matrix}$$

So we have $\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ which checks out.

Problem 4. (6 points)Find all values of h , such that the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ h \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

are linearly independent.

We have to find all values of h for which the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & h & 0 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{pmatrix} \text{ has a pivot in every column.}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & h & 0 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{pmatrix} \xrightarrow{-I} \begin{pmatrix} 1 & 0 & -1 \\ 0 & h & 1 \\ 0 & 3 & 3 \\ 0 & 5 & 5 \end{pmatrix} \xleftarrow{\text{we cannot make } h=1 \text{ because that would mean dividing by } h.} \text{ better to swap up a row from below to create a pivot in the position with } h.$$

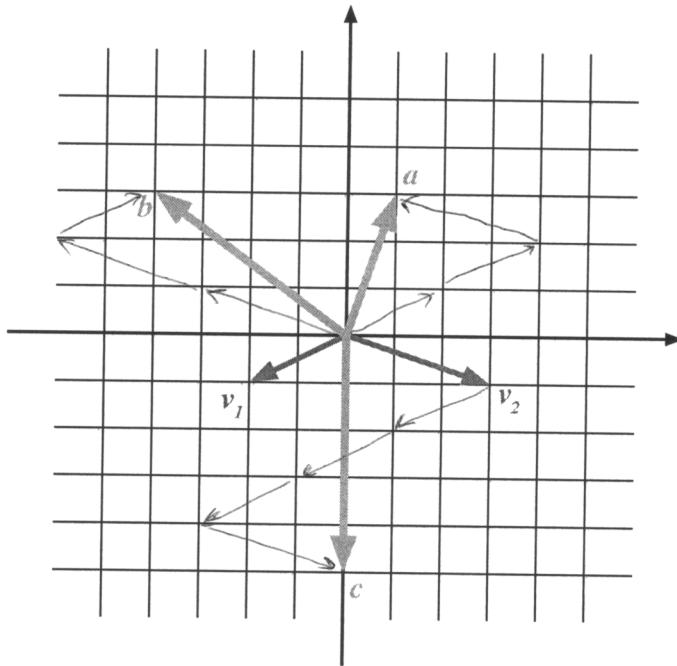
$$\xrightarrow{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 3 \\ 0 & h & 1 \\ 0 & 5 & 5 \end{pmatrix} \cdot \frac{1}{3}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & h & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{-h \text{ II}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1-h \\ 0 & 0 & 0 \end{pmatrix}$$

Now we see that there are 3 pivots if and only if $h \neq 1$.So the three vectors are linearly independent if and only if $\boxed{h \neq 1}$.In fact if $h=1$ then the rref of our matrix is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with general solution $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Taking, for example $x_3=1$ gives the linear relation with weights $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ among the three vectors. Check:

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 5. (5 points)

Referring to the sketch, write the vectors \vec{a} , \vec{b} and \vec{c} as linear combinations of \vec{v}_1 and \vec{v}_2 .



use head-to-tail addition.

$$\vec{a} = -\vec{v}_1 - \vec{v}_2 - \vec{v}_2 = -2\vec{v}_1 - \vec{v}_2$$

$$\vec{b} = -\vec{v}_2 - \vec{v}_2 - \vec{v}_1 = -\vec{v}_1 - 2\vec{v}_2$$

$$\vec{c} = \vec{v}_2 + \vec{v}_1 + \vec{v}_1 + \vec{v}_1 + \vec{v}_2 = 3\vec{v}_1 + 2\vec{v}_2$$

Problem 6. (6 points)

For each transformation decide whether or not it is linear. If it is linear, find its matrix.

(a) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y-x \\ x \end{pmatrix}$.

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y-x \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

So T is a matrix transformation, therefore, T is linear.

matrix of T : $[T] = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$ which is 3×2 as it should be for $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

(b) $S : \mathbb{R}^4 \rightarrow \mathbb{R}^4$, which multiplies every vector by the scalar 5.

S is linear: $S(x_1 \vec{u}_1 + x_2 \vec{u}_2) = 5(x_1 \vec{u}_1 + x_2 \vec{u}_2) = x_1 5\vec{u}_1 + x_2 5\vec{u}_2$
 $= x_1 S(\vec{u}_1) + x_2 S(\vec{u}_2)$.

$S(\vec{e}_i) = 5\vec{e}_i$ for $i=1,2,3,4$. So the matrix of S is

$$[S] = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

(c) $U : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+1 \\ x-y-1 \end{pmatrix}$

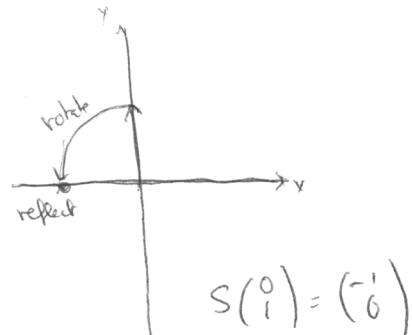
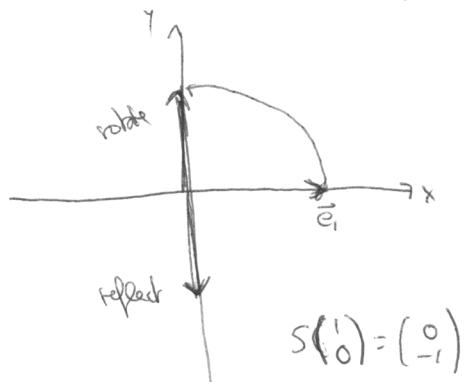
$$U \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ so } U(\vec{0}) \neq \vec{0}, U \text{ cannot be linear.}$$

Problem 7. (6 points)

Consider the linear transformations S , and T , both with domain \mathbb{R}^2 and codomain \mathbb{R}^2 .

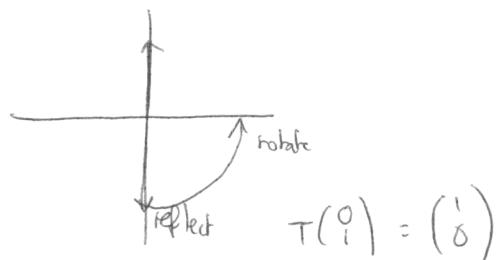
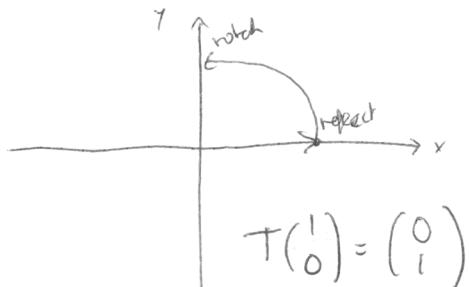
- (a) The transformation S first rotates every vector by an angle of $\pi/2$ counterclockwise about the origin, and then reflects the resulting vector across the x -axis. Find the matrix of S .

$$[S] = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

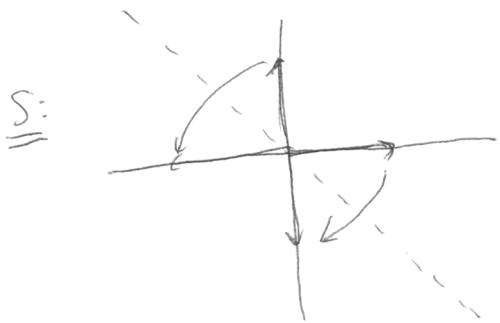


- (b) The transformation T first reflects every vector across the x -axis, and then rotates the resulting vector counterclockwise about the origin by an angle of $\pi/2$. Find the matrix of T .

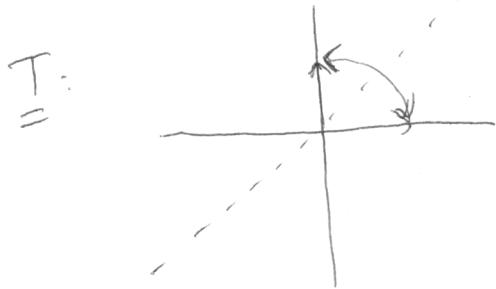
$$[T] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



(c) Describe the transformation S as a single geometric operation. Do the same for T .



this is reflection across the line $x+y=0$.



this is reflection across the line $x=y$.