

Problem 1.  $x=0$  (it is under a pivot)

$z=1$  (there must be a pivot in Row II, because there is one in Row III)

$y = \text{anything}$ .

$u=0$   
 $w=0$  } they are above a pivot

$\begin{pmatrix} 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$  is in rref.

Problem 2.  $(1 \ 0 \ 0)$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \end{pmatrix}$  ←  $*$  can be anything

$\begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$  ←

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}$  } any number of rows containing only 0s.

Problem 3. Find out if  $x_1 \vec{u}_1 + x_2 \vec{u}_2 = \vec{v}$  has any solutions for  $x_1, x_2$ :

$$\begin{pmatrix} 1 & 1 \\ \vec{u}_1 & \vec{u}_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{v}$$

in other words, is the system with augmented matrix  $\left( \begin{array}{cc|c} 2 & 13 & k \\ 2 & 1 & 2 \\ 1 & 8 & k \end{array} \right)$  consistent?

$$\left( \begin{array}{cc|c} 2 & 13 & k \\ 2 & 1 & 2 \\ 1 & 8 & k \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 2 & 13 & k \\ 0 & -12 & 2-k \\ 0 & 3 & k \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 2 & 13 & k \\ 0 & 3 & k \\ 0 & 0 & 2+3k \end{array} \right) \text{ need } 2+3k=0 \text{ for consistent.}$$

for  $k = -2/3$  the vector  $\vec{v}$  is a linear combination of  $\vec{u}_1$  and  $\vec{u}_2$ . Otherwise, not.

Problem 4. Does  $\begin{pmatrix} 1 & 4 & 3 \\ 3 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$  have a non-trivial solution?

$$\begin{pmatrix} 1 & 4 & 3 \\ 3 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{matrix} \\ -3I \\ -2I \end{matrix} \text{ (no need for augmentation column in homogeneous case)}$$

$$\rightarrow \begin{pmatrix} 1 & 4 & 3 \\ 0 & -13 & -9 \\ 0 & -6 & -5 \end{pmatrix} \begin{matrix} \\ -2III \\ \end{matrix} \rightarrow \begin{pmatrix} 1 & 4 & 3 \\ 0 & -1 & 1 \\ 0 & -6 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & -1 \\ 0 & 6 & 5 \end{pmatrix} \begin{matrix} \\ \\ -6I \end{matrix} \rightarrow \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 11 \end{pmatrix}$$

a pivot in every column (no free variables, no non-trivial solution) so the vectors are linearly independent.

Problem 5. The equation of a plane through the origin will be  $ax + by + cz = 0$ .

$\vec{u}_1$  satisfies this equation:

$$a + b + 2c = 0 \leftarrow \text{from } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$\vec{u}_2$  satisfies this equation:

$$-2b + 4c = 0 \leftarrow \text{from } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$$

Solve for  $a, b, c$ :

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ a & b & c \end{matrix}$

$$a = -4c$$

$$b = 2c$$

$$c = \text{free}$$

any non-trivial solution will do, so set  $c=1$  to get

$$a = -4$$

$$b = 2$$

$$c = 1$$

An equation is  $\boxed{-4x + 2y + z = 0}$

other possibilities:  $4x - 2y - z = 0$

$$8x - 4y - 2z = 0$$

etc.

Problem 6. general solution in parametric vector form:  $(2 \ -3 \ 6) \rightarrow (1 \ -3/2 \ 3)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/2 y - 3z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 3/2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \quad \text{So } \begin{pmatrix} 3/2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \text{ span the plane } 2x - 3y + 6z = 0.$$

because every solution is a linear combination of these two vectors.

Problem 7. Is the system with augmented matrix  $\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 3 \end{array}\right)$  consistent?

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 3 \end{array}\right) +I \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 2 & 4 \end{array}\right) +2II \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{array}\right) \text{ is consistent.}$$

↑ pivot in augmentation col.

No,  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$  is not in  $\text{span}\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right)$ .

Problem 8.  $\left(\begin{array}{ccc|c} 2 & -4 & 1 & k \\ 1 & -2 & 2 & k \\ 1 & -2 & 1 & 2k \\ 1 & -2 & 1 & k \end{array}\right)$  ← homogeneous system: no need for augment. col.  
 swap to avoid fractions.

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & k \\ 2 & -4 & 1 & k \\ 1 & -2 & 1 & 2k \\ 1 & -2 & 1 & k \end{array}\right) \begin{array}{l} -2I \\ -I \\ -I \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 2 & k \\ 0 & 0 & -3 & -k \\ 0 & 0 & -1 & k \\ 0 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} \\ \\ +II \\ +3II \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & -k \end{array}\right) +III \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 2 & k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k \end{array}\right) -2II - III \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k \end{array}\right)$$

Case 1  $k \neq 0$  then rref is  $\begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ .  $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ .

Case 2  $k=0$  then rref is  $\begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ .  $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ .

Problem 9.  $\left(\begin{array}{cc} 1 & 2 \\ k & 5k-1 \end{array}\right) -kI \rightarrow \left(\begin{array}{cc} 1 & 2 \\ 0 & 5k-1-2k \end{array}\right) = \left(\begin{array}{cc} 1 & 2 \\ 0 & 3k-1 \end{array}\right)$  the columns of

the matrix are linearly independent if rref has a pivot in every column. So we need  $3k \neq 1$  or  $k \neq 1/3$ . If  $k \neq 1/3$  the vectors  $\begin{pmatrix} 1 \\ k \end{pmatrix}, \begin{pmatrix} 2 \\ 5k-1 \end{pmatrix}$  are linearly independent.

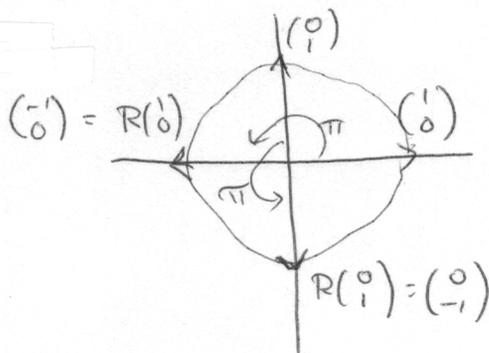
(If  $k=1/3$ , they are dependent:  $\begin{pmatrix} 1 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2/3 \end{pmatrix}$ .)

### Problem 10

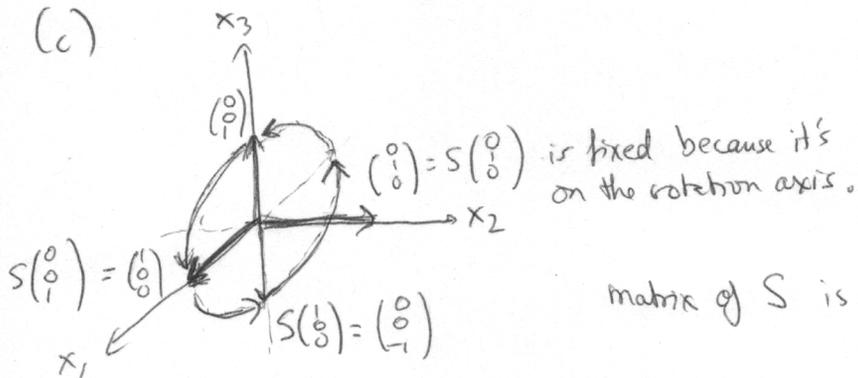
$$(a) \quad T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a-b \\ a+c-d \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \end{pmatrix} + d \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

matrix of  $T$  is  $[T] = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix}$ .

(b) matrix of  $R$  is  $[R] = \begin{pmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ .



(c)



matrix of  $S$  is  $[S] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ .

## Problem 11.

(a) The domain of  $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$  is  $\mathbb{R}^3$ , the codomain is  $\mathbb{R}^2$ .

(b) Is  $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$  onto?  $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  Not a pivot in every row

So  $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$  is not onto. (The range is  $\text{span}(\begin{pmatrix} 1 \\ 2 \end{pmatrix})$  which is a line through the origin in  $\mathbb{R}^2$ , which is different from the codomain.)

(c) The domain of  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$  is  $\mathbb{R}^2$ , the codomain is  $\mathbb{R}^3$ .

(d) Is  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$  one-to-one?  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \xrightarrow{-3I} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}$

there is a pivot in every column so yes  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$  is one-to-one.

## Problem 12

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$B \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

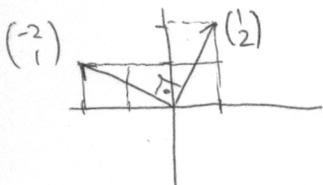
$$A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$B \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$[B \circ A] = \begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}$$

Problem 13  $x+2y=0$ .  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$  is a vector on this line,  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  is a vector perpendicular to the line. So  $T \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$T \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



If the matrix is  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  we have  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -2a + b &= -2 \\ -2c + d &= 1 \\ a + 2b &= 0 \\ c + 2d &= 0 \end{aligned}$$

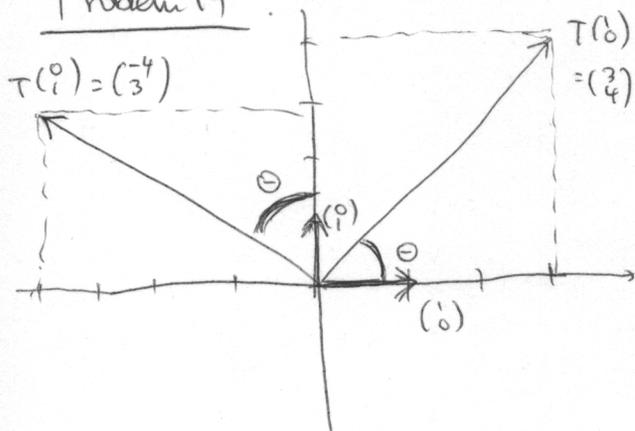
$$\left( \begin{array}{cccc|c} -2 & 1 & 0 & 0 & -2 \\ 0 & 0 & -2 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ -2 & 1 & 0 & 0 & -2 \\ 0 & 0 & -2 & 1 & 1 \end{array} \right) \xrightarrow{\substack{+2I \\ +2II}} \left( \begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 5 & 0 & 0 & -2 \\ 0 & 0 & 0 & 5 & 1 \end{array} \right) \leftarrow \begin{array}{l} \\ \\ \end{array} \left( \begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2/5 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1/5 \end{array} \right) \begin{array}{l} -2II \\ -2IV \end{array}$$

$$\rightarrow \left( \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4/5 \\ 0 & 1 & 0 & 0 & -2/5 \\ 0 & 0 & 1 & 0 & -2/5 \\ 0 & 0 & 0 & 1 & 1/5 \end{array} \right) \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 4/5 \\ -2/5 \\ -2/5 \\ 1/5 \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{pmatrix}$$

(There was a misprint: it was supposed to be  $\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$ .)

### Problem 14

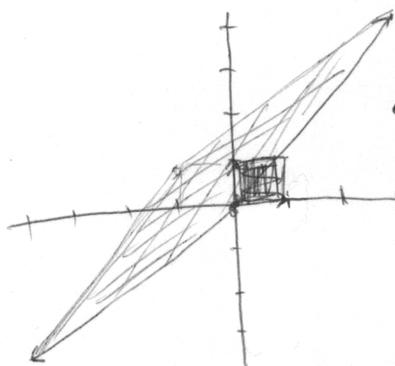


From the sketch we see that both  $\vec{e}_1, \vec{e}_2$  are rotated by the same angle, and rescaled by the same scalar. The scaling factor is

$$5 = \sqrt{3^2 + 4^2}, \text{ the angle is } \theta = \arccos 3/5.$$

$$\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} = 5 \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ where } \theta = \arccos 3/5.$$

the matrix  $\begin{pmatrix} 3 & -4 \\ 4 & -3 \end{pmatrix}$ :



← this is harder to describe geometrically, Sorry!  $\nabla$   
the basic square gets rotated by  $T_2$  and stretched out sideways.

Problem 15  $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  so the span of the columns of the matrix is equal to  $\text{span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$  which is a plane through the origin in  $\mathbb{R}^3$ .

