

1. (8 points) Solve the following system of linear equations. Write the solution in a parametric vector form. Show your work.

$$x_2 + x_4 = 1$$

$$-2x_1 + 2x_2 + 2x_3 + x_4 = 0$$

$$4x_1 - 4x_2 - 4x_3 - 2x_4 = 0$$

$$\rightarrow \left(\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 1 \\ -2 & 2 & 2 & 1 & 0 \\ 4 & -4 & -4 & -2 & 0 \end{array} \right) \sim \left(\begin{array}{cccc|c} -2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 4 & -4 & -4 & -2 & 0 \end{array} \right)$$

$$R_3 + 2R_1 \left(\begin{array}{cccc|c} -2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \text{ REF} \quad R_1 - 2R_2 \left(\begin{array}{cccc|c} -2 & 0 & 2 & -1 & -2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$-\frac{1}{2}R_1 \left(\begin{array}{cccc|c} 1 & 0 & -1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \text{ RREF} \quad \text{free var}$$

$$x_1 = 1 + x_3 - \frac{x_4}{2}$$

$$x_2 = 1 - x_4$$

$$x \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + x_3 - \frac{x_4}{2} \\ 1 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

4 pts

2. (16 points) For each of the following linear maps $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $L: \mathbb{R}^p \rightarrow \mathbb{R}^q$, answer to these questions: (a) determine the domain, codomain and find the standard matrix. (b) Determine if the linear transformation is one-to-one and onto its codomain. (c) Find the range. (2 pts)

i) $I_n = \begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} x_2 + 3x_3 \\ x_1 + 2x_2 + 6x_3 \end{bmatrix}$ $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ domain = \mathbb{R}^3 , codomain = \mathbb{R}^2 (2 pts)

$T(e_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $T(e_2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $T(e_3) = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 6 \end{pmatrix}$ (2 pts)

OR: $T(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_2 + \begin{pmatrix} 3 \\ 6 \end{pmatrix} x_3 = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A x$

$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \end{pmatrix}$ one pivot at each row \Rightarrow onto \mathbb{R}^2 \checkmark
 REF one free variable \Rightarrow NOT one-to-one (2 pts)

Range of $T = \mathbb{R}^2$ since T is onto \mathbb{R}^2 (1 pt)

ii) $I_2 = \begin{pmatrix} e_1 & e_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ $L \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $L \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ domain = \mathbb{R}^2 , codomain = \mathbb{R}^2 (2 pts)

L is linear map: $L \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -L \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2L \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -L(e_1) + 2L(e_2) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$
 $L \begin{pmatrix} 1 \\ -1 \end{pmatrix} = L(e_1) - L(e_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

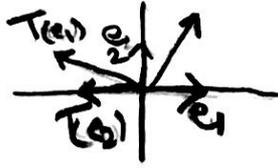
$\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} L(e_1) \\ L(e_2) \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 & | & 3 \\ 1 & -1 & | & 0 \end{pmatrix} \xrightarrow{R_2, R_1} \begin{pmatrix} -1 & 2 & | & 3 \\ 0 & 1 & | & 3 \end{pmatrix} \xrightarrow{-R_1} \begin{pmatrix} 1 & -2 & | & -3 \\ 0 & 1 & | & 3 \end{pmatrix} \sim$
 $\sim \begin{pmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 3 \end{pmatrix} \Rightarrow L(e_1) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $L(e_2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow A = [L(e_1) \ L(e_2)] = \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix}$ (4 pts)

$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$ REF one pivot at each row \Rightarrow onto \mathbb{R}^2
 one pivot at each col. \Rightarrow one-to-one (2 pts)

Range of $L = \mathbb{R}^2$ since L is onto \mathbb{R}^2 (1 pt)

Alternative solut. ii) a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\begin{cases} -a + 2b = 3 \\ -c + 2d = 0 \\ a - b = -1 \\ c - d = 1 \end{cases} \Rightarrow \begin{pmatrix} -1 & 2 & 0 & 0 & | & 3 \\ 0 & -1 & -1 & 2 & | & 0 \\ 1 & -1 & 0 & 0 & | & -1 \\ 0 & 0 & 1 & -1 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 1 & | & 1 \end{pmatrix} \Rightarrow \begin{matrix} a=1 \\ b=2 \\ c=2 \\ d=1 \end{matrix} \Rightarrow A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$



3. (8 points) Find the standard matrix for the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which first performs a vertical shear with a factor of 2. Second, it rotates points around the origin counterclockwise by an angle of $\pi/2$. Show your work.

e_1, e_2
 $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow[\text{shear } c=2]{\text{vertical}} \begin{bmatrix} 1 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{\text{rot}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = T(e_1)$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix} = T(e_2)$

$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$

8pts

or: vertical shear: $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ factor $c=2$ rot $\pi/2$: $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$T(m) = BAX = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$

4. (6 points) Let A be the standard matrix corresponding to the linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$. Decide if the following arguments are true or false. No need to explain why.

- (a) • If A has m pivots, the range of T is \mathbb{R}^m . *true*
- (b) • If the columns of A are linearly independent, then they will always span \mathbb{R}^m . *false*
- (c) • If $m = n$ and there is one pivot in each row, the columns of A are linearly independent. *true*
- (d) • If $m \neq n$, the linear system $Ax = b$, $b \in \mathbb{R}^m$ will never have a unique solution. *false*
- (e) • If $n = m$, and $B \in \mathbb{R}^{n \times n}$ then we have always $AB = BA$. *false*
- (f) • Transpose of matrix A has m columns and n rows. *true*

(a): one pivot each row $\Rightarrow A$ span $\mathbb{R}^m \Rightarrow \text{range}(T) \subseteq \mathbb{R}^m$ (2 pivots)

(b): counterexample: $m=3, n=2 \Rightarrow$ cols of A are linearly indep. but it does not span \mathbb{R}^3

(c): nb pivots = nb. rows = $m = n =$ nb. cols \Rightarrow one pivot at each col \Rightarrow cols. are lin. indep.

(d): ex: $m=3, n=2, r=2: A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow m \neq n$ unique solution.

(e): AB and BA are both $n \times n$ matrices, but are not necessarily the same.

(f): $A: m$ cols. m rows $\Rightarrow A^T: m$ cols. and n rows ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

5. (8 points) Find the inverse of matrix A. Use the algorithm introduced during the course (with row reduction). Show your work.

$$[A | I_n] \sim [I_n | A^{-1}] \quad A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{R_2+3R_1 \\ R_3-2R_1}} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{R_3+3R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & -2 & 1 & 0 & 0 \\ 0 & \boxed{1} & -2 & 3 & 1 & 0 \\ 0 & 0 & \boxed{2} & 7 & 3 & 1 \end{array} \right] \xrightarrow{\substack{R_1+R_3 \\ R_2+R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array} \right]$$

$$\xrightarrow{R_3/2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array} \right] \Rightarrow A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 3.5 & 1.5 & 0.5 \end{bmatrix}$$

8pts

6. (2 points) $\{a_1\} \in \mathbb{R}^m$ is a set with only one vector. Is this vector set always linearly independent? NO

If no, in which condition for a_1 the set $\{a_1\} \in \mathbb{R}^m$ will be linearly independent?

if $a_1 = 0$, $\forall x \in \mathbb{R}^m$ the homog. system $a_1 x = 0$ has

infinitely many solut. $\therefore \overbrace{0x=0}^{\text{always true}} \Rightarrow$ 2pts

\hookrightarrow If $a_1 \neq 0 \Rightarrow a_1 x = 0$ has only the trivial solution \Rightarrow the condition is $a_1 \neq 0$