Erratum on page 68 of lecture notes

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Example 13.8. Consider the plane $x - y + z = 0$.

(a) Find the 3×3 matrix T_1 which represents projection of $\mathbb{R}^3$ onto a vector orthogonal to this plane.

Solution:

In general, an orthogonal vector to any plane in $\mathbb{R}^3$ written in the form $ax+by+cz = d$is given as: $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Thus, in this particular example, the orthogonal vector is:

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

We are asked to find a linear transformation $T_2(x) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ which projects any point on the subspace spanned by u_1, that is a line in $\mathbb{R}^3$. Let's call this line the subspace H, and a basis for it is $\mathcal{U} = \{u_1\}$. Obviously, this is an orthogonal basis (Why?). As we learned previously about finding the standard matrix of a linear transformation, we need to find the "action" of the linear map on the columns of identity matrix. Thanks to the orthogonality of this basis, from Theorem 29 in the lecture notes (or Theorem 8 in the textbook) we have:

$$T_1(e_1) = \text{proj}_H e_1 = \frac{u_1 \cdot e_1}{u_1 \cdot u_1} u_1 = \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$T_1(e_2) = \text{proj}_H e_2 = \frac{u_1 \cdot e_2}{u_1 \cdot u_1} u_1 = \frac{-1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$T_1(e_3) = \text{proj}_H e_3 = \frac{u_1 \cdot e_3}{u_1 \cdot u_1} u_1 = \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$A_1 = [T_1(e_1) \qquad T_1(e_2) \qquad T_1(e_3)] = \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

(b) Let $a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Find vectors v and w such that $a = v + w$, where v is in the plane and w is perpendicular to the plane.

Solution:

$$w = \text{proj}_H a = \frac{a \cdot u_1}{u_1 \cdot u_1} u_1 = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$v = a - w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \\ \frac{2}{3} \end{pmatrix}$$

(c) Find the 3×3 matrix T_2 which represents projection of $\mathbb{R}^3$ onto this plane.

Solution:

Similar to (a), we want to find a the "action" of the linear map to the columns of the identity matrix. As we did for vector a in part (b), we can find the projection of e_1, e_2, e_3 on the plane $x - y + z = 0$ by:

$$T_2(e_1) = e_1 - T_1(e_1)$$

$$T_2(e_2) = e_2 - T_1(e_2)$$

$$T_2(e_3) = e_3 - T_1(e_3)$$

$$A_2 = [T_2(e_1) \qquad T_2(e_2) \qquad T_2(e_3)] = I_3 - A_1 = \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$