Quiz 9 - Math221 - Sec 203 Mar 31, 2017

Name: _____ Student number: _____

No books. No notes. No calculator. No electronic device of any kind.

1. (*8 points*) Let $B = \{u_1, u_2, u_3\}$ be a basis for the subspace $H \in \mathbb{R}^4$.

$$u_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$

(a) Show that B is an orthogonal basis.

$u_1 . u_2 = 0 - 4 + 0 + 4 = 0$

$u_1 . u_3 = \qquad = 0$

$u_2 . u_3 = \qquad = 0$

(b) Find the coordinate of $v_1 = \begin{pmatrix} 3 \\ -4 \\ 1 \\ -1 \end{pmatrix}$ relative to the basis B.

$$v = \frac{v . u_1}{\underbrace{u_1 . u_1}} u_1 + \frac{v . u_2}{\underbrace{u_2 . u_2}_{=1}} u_2 + \frac{v . u_3}{\underbrace{u_3 . u_3}_{=1}} u_3$$

$$= \frac{0 - 8 + 1 - 2}{0 4 + 1 + 4} = -1$$

$$v = -u_1 + u_2 + u_3 \Rightarrow [v]_B = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

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(c) Find a vector u_4 which is orthogonal to all of u_1, u_2 and u_3.

$$A = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \qquad u_4 \in Col(A)^{\perp} = Nul(A^T) \Rightarrow A^T u_4 = 0$$

$$A^T = \begin{pmatrix} 0 & 2 & 1 & 2 \\ 1 & -2 & 0 & 2 \\ 2 & 0 & 2 & -1 \end{pmatrix} \overset{r.r}{\sim} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow x = x_3 \begin{pmatrix} -1 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} \quad x_3 \in \mathbb{R}$$

~~exam~~ $Nul(A)^T = x_3 \begin{pmatrix} -1 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}$; one example $u_4 = \begin{pmatrix} -1 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}$

Check: $u_4 \cdot u_3 = \begin{pmatrix} -1 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix} = 0$ ✓ $\quad u_4 \cdot u_1 = \begin{pmatrix} -1 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix} = 0$ ✓

$u_4 \cdot u_2 = \begin{pmatrix} -1 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix} = 0$ ✓

2. (*6 points*) Consider the following matrix A and vector b:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

Find the least squares solution $\hat{x}$ for the linear system $Ax = b$. Using this, find a linear function $y = \alpha x + \beta$ whose graph best fits the points $(-1, 0), (0, 1), (1, 2)$ and $(2, 2)$ in $\mathbb{R}^2$.

$$\begin{aligned} 0 &= -\alpha + \beta \\ 1 &= 0\alpha + \beta \\ 2 &= 1\alpha + \beta \\ 2 &= 2\alpha + \beta \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}}_{b} \Rightarrow \text{inconsistent, we solve } A^T A \hat{x} = A^T b$$

$$AA^T = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \qquad A^T b = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 6 & 2 & 6 \\ 2 & 4 & 5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1/3 & 1 \\ 1 & 2 & 5/2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1/3 & 1 \\ 0 & 5/3 & 3/2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0.7 \\ 0 & 1 & 0.9 \end{array}\right) \quad \begin{array}{l} \alpha = 0.7 \\ \beta = 0.9 \end{array}$$

$$\Rightarrow y = 0.7x + 0.9$$

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