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SOME LIMIT THEOREMS FOR A CLASS OF NETWORK PROBLEMS AS RELATED TO FINITE MARKOV CHAINS

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Abstract

This paper is concerned with a class of dynamic network flow problems in which the amount of flow leaving node i in one time period for node j is the fraction p_{ij} of the total amount of flow which arrived at node i during the previous time period. The fraction p_{ij} whose sum over j equals unity may be interpreted as the transition probability of a finite Markov chain in that the unit flow in state i will move to state j with probability p_{ij} during the next period of time. The conservation equations for this class of flows are derived, and the limiting behavior of the flows in the network as related to the properties of the fractions p_{ij} are discussed.

DYNAMIC NETWORK FLOW; FINITE MARKOV CHAIN; TRANSITION PROBABILITIES; NON-NEGATIVE MATRICES; APERIODICITY; IRREDUCIBILITY

1. Introduction

We consider a class of dynamic network flow problems in which the amount of flow leaving one node, say i, in one time period for another node, say j, is a certain fraction of the total amount of flow which arrived at node i during the previous time period. Assuming that there is some flow into the network in each time period from outside the network, our problem is to find the limiting behavior of the flows in the network as time goes to infinity. In the case where the total amount of flow in the network increases monotonically as time increases, we are interested in determining the limiting behavior of the fraction of the amount of flow in each arc to the total amount of flow in the network.

Related work on stochastic networks includes models of general queue networks [1], pp. 215–218, and models of stochastic shortest paths [10], pp. 740–741. The model in this paper, however, is most closely related to Markovian population models. (See, for example, [4], pp. 469–490, [7], [8] and [9].)

2. Formulation of the problem

Let us consider a network $G(N,A)$ which consists of a collection N of nodes $\{1,2,\cdots,H\}$, together with a subset A of ordered pairs (i,j) of nodes from N. Let us denote by 0 the source node which supplies a certain amount of flow to each

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node. We assume that we are given p_{ij}, a number associated with each arc $(i,j) \in N$ such that

$$\sum_{j=1}^{H} p_{ij} = 1, \; p_{ij} \geqq 0. \tag{1}$$

Let $f(i,j;t)$ be the amount of flow that leaves i along (i,j) at time t and subsequently arrives at j at $t+1$. Then the conservation equations for the class of dynamic network flows we will study in this paper are recursively defined by

$$\left[\sum_{i=0}^{H} f(i,j;t)\right] p_{jk} = f(j,k;t+1), \quad j,k \in N, \qquad t = 0,1,2,\cdots, \tag{2}$$

where the $f(i,j;0) \geqq 0$ for $i, j \in N$ are assumed given, $f(0,j;t) \geqq 0$ is the amount of flow that leaves the source at time t, and subsequently arrives at j at $t+1$. The $f(0,j;t)$ are also assumed given for all $j \in N$. We consider the following three cases of $f(0,j;t)$.

Case 1:

$$f(0,j;t) = b_j, \qquad j \in N, \; t = 0,1,2,\cdots; \tag{3}$$

Case 2:

$$f(0,j;t) = a_j \sum_{i=1}^{H} f(i,j;t), \qquad j \in N, \; t = 1,2,\cdots; \tag{4}$$

and

Case 3:

$$f(0,j;t) = \sum_{i=1}^{H} m_{ij} \sum_{k=1}^{H} f(k,i;t), \qquad j \in N, \, t = 1,2,\cdots; \tag{5}$$

where $b_j \geqq 0$, $a_j \geqq 0$ and $m_{ij} \geqq 0$ are given constants.

While in Case 1 the amount of flow coming from the source to each node in N is assumed constant, the incoming flows are assumed to be linear functions of the amount of flow in the network G in Cases 2 and 3. In Case 2 the amount of flow from the source to node j is equal to the amount of flow from nodes in N to node j multiplied by a_j. In Case 3 the amount of flow from the source to node j is proportional to the weighted total of the amount of flow in the network G. If $m_{ij} = m_j$ in Case 3, $f(0,j;t)$ is equal to the total amount of flow in the network G multiplied by m_j.

Our problem is to investigate the limiting behavior of $f(i,j;t)$ as t goes to infinity. In particular, we are interested in determining under certain conditions the following quantities: (i)

$$f(i,j;\infty) \triangleq \lim_{t\to\infty} f(i,j;t) \tag{6}$$

for Case 1, and (ii)

(7) $$z_{ij} \triangleq \lim_{t\to\infty} f(i,j;t) \Big/ \sum_{k=1}^{H} \sum_{l=1}^{H} f(k,l;t)$$

for Cases 1, 2 and 3.

It is noted that the p_{ij} defined by (1) may be interpreted as transition probabilities. That is, p_{ij} could be the probability that the unit amount of flow in state i in the present time period will be found in state j in the next time period. In this case $f(i,j;t)$ would be the expected amount of flow leaving i at time t, and hence arriving at j at $t+1$, and our problem would be considered a class of network flows defined on a finite Markov chain characterized by p_{ij}. This is the interpretation adhered to in the following sections of this paper.

It is also noted that summing both sides of (2) over k gives us

(8) $$\sum_{i=0}^{H} f(i,j;t) = \sum_{k=1}^{H} f(j,k;t+1)$$

where these equations are the conservation equations for a special case of the general dynamic flow problems studied in [2].

3. Limiting behavior of $f(i,j;t)$

We first consider a network $G(N,A)$ in which the node H is an absorbing state, that is, $p_{Hj} = 1$ if $j = H$ and 0 otherwise. Hence, any flow which has arrived at H will remain there forever. Then we have the following theorem.

Theorem 1. Let H be an absorbing state. Assume that

(3) $$f(0,j;t) = b_j, \quad j \in N, \; t = 0,1,2,\cdots$$

holds, and that all eigenvalues of an $(H-1)$ by $(H-1)$ matrix

$$P_1 \triangleq \{p_{ij};\, 1 \leqq i,j \leqq H-1\}$$

are less than one in absolute value. Then $\lim_{t\to\infty} f(i,k;t)$ exists for all $i \in \{1,2,\cdots,H-1\}$ and $k \in N$, and satisfies

(9) $$\left[b_j + \sum_{i=1}^{H-1} f(i,j;\infty)\right] p_{jk} = f(j,k;\infty), \qquad j \in \{1,2,\cdots,H-1\},\; k \in N.$$

Remark.[1] The condition that P_1 has no eigenvalue on the unit circle is related to the non-stochastic character of P_1. See, for example, [5], pp. 94–103 and Theorem 3.2 in particular, and [6].

[1] The author is indebted to the referee for this remark.

Proof. Let $x_j(t)$ be the amount of flow arriving at j at time $t+1$; that is,

$$(10) \qquad x_j(t) = \sum_{i=1}^{H-1} f(i,j;t), \quad j \in \{1,2,\cdots,H-1\}.$$

Noting that $f(H,j;t) = 0$ for $j \in \{1,2,\cdots,H-1\}$ and all t, it follows from (2), (3) and (10) that

$$(11) \qquad d_k + \sum_{j=1}^{H-1} p_{jk} x_j(t) = x_k(t+1), \quad k \in \{1,2,\cdots,H-1\},$$

where

$$(12) \qquad d_k = \sum_{j=1}^{H-1} b_j p_{jk}.$$

Letting[2] $x(t) = (x_1(t), x_2(t), \cdots, x_{H-1}(t))'$ and $d = (d_1, d_2, \cdots, d_{H-1})'$, (11) is written in the vector form:

$$(13) \qquad x(t+1) = P_1' x(t) + d.$$

The general solution of (13) is given by

$$(14) \qquad x(t) = (P_1')^t x(0) + \left(\sum_{\tau=1}^{t-1} (P')^\tau + I \right) d,$$

where I is the $(H-1)$ by $(H-1)$ unit matrix. By assumption we have $\lim_{t\to\infty} (P_1')^t = 0$ and

$$\lim_{t\to\infty} \left(\sum_{\tau=1}^{t} (P_1')^\tau + I \right) = (I - P_1')^{-1}.$$

(See, for example, [3].) Hence, by letting t go to infinity on both sides of (14) we have

$$(15) \qquad x(\infty) \underline{\underline{\Delta}} \lim_{t\to\infty} x(t) = (I - P_1')^{-1} d$$

or

$$(16) \qquad x(\infty) = d + P_1' x(\infty).$$

Since $x(\infty)$ exists and since $f(i,j;t) \geqq 0$ for all i, j and t, the $f(i,j;\infty)$ exist for all $i,j \in \{1,2,\cdots,H-1\}$. Since we have from (2), (3) and $f(H,j;t) = 0$ for $j \in \{1,2,\cdots,H-1\}$ that

$$(17) \qquad \left[b_j + \sum_{i=1}^{H-1} f(i,j;t) \right] p_{jk} = f(j,k;t+1),$$

$$j \in \{1,2,\cdots,H-1\}, \qquad k \in N,$$

(9) follows by letting t go to infinity on both sides of (17).

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[2] The prime denotes the transpose of a vector or a matrix.

Example. Let $H = 4$, $b_1 = 1$, $b_2 = b_3 = 0$, $p_{12} = p_{32} = p_{34} = 0.5$, $p_{13}=0.4$, $p_{14}=0.1$, $p_{24}=1$ and let all other p_{ij} be zero. Then the equations (8) are solved to give: $f(1,2;\infty) = 0.5$, $f(1,3;\infty) = 0.4$, $f(3,2;\infty) = f(3,4;\infty) = 0.2$, $f(2,4;\infty) = 0.7$, $f(1,4;\infty) = 0.1$, and all other $f(i,j;t)$ for $i = 1, 2, 3$ and $j = 1,2,3,4$ are zero.

We now turn our attention to determining the fraction distribution z_{ij} defined by (7) of the flows over the arcs of the network. We need the following definitions and theorem concerning non-negative matrices. (See [3] for a general treatment of non-negative matrices, and see [5] and [6] for the implications of properties of non-negative matrices in finite Markov chains.)

Definition. A non-negative matrix L is reducible if there is a permutation that can place it in the form

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B and D are square matrices; otherwise it is called irreducible. (An irreducible Markov transition probability matrix corresponds to a process in which all states communicate. See [5] for a further discussion of irreducibility.)

Definition. A non-negative matrix L is aperiodic (or acyclic or primitive) if there exists a positive integer $p\,(<\infty)$ such that $L^p > 0$.

Theorem 2. Let L be an H by H aperiodic and irreducible matrix and let $y(t) = (y_1(t), y_2(t), \cdots, y_H(t))'$ be a solution to

$$y(t+1) = Ly(t) \tag{18}$$

where $y_i(0)$ is assumed given such that $y_i(0) \geqq 0$ for all i and such that there exists at least one k for which $y_k(0) > 0$. Then there exists a $z = (z_1, z_2, \cdots, z_H)'$ which is independent of $y(0)$ and such that

$$z = \lim_{t\to\infty} \sum_{i=1}^{n} y(t)/y_i(t), \tag{19}$$

and z satisfies the following system uniquely:

$$\left[\lambda z = Lz,\ z \geqq 0,\ \sum_{i=1}^{H} z_i = 1,\ \lambda \geqq 0\right], \tag{20}$$

where λ is a scalar. Furthermore, we always have $z > 0$ and $\lambda > 0$.

It is shown that the z and λ which satisfy (20) are unique, and that λ is the maximum eigenvalue of L and z is the corresponding eigenvector. The proof of this theorem is found in [3].

The following theorem is concerned with the z_{ij} for Case 2 as defined by (4).

Theorem 3. Let $f(0,j;t)$ be given by (4), and let L be an H by H matrix

$$L = \{p_{ij}(1 + a_i);\ i,j \in N\}' \tag{21}$$

where $p_{ij}(1 + a_i)$ is the (j, i)th element of L. If L is aperiodic and irreducible, then there exist z_{ij} defined by (7) for $i, j \in N$, and these z_{ij} are given by

$$z_{jk} = z_j(1 + a_j)\, p_{jk}, \tag{22}$$

where z is the solution to (20), and

$$z_j = \sum_{i=1}^{H} z_{ij}. \tag{23}$$

Proof. We have from (2) and (4) that

$$\left[(1 + a_j) \sum_{i=1}^{H} f(i,j;t)\right] p_{jk} = f(j,k;t+1). \tag{24}$$

Let

$$y_j(t) = \sum_{i=1}^{H} f(i,j;t), \tag{25}$$

and let $y(t) = (y_1(t), y_2(t), \cdots, y_H(t))'$. Then summing both sides of (24) over j gives

$$y_k(t+1) = \sum_{j=1}^{H} (1 + a_j)\, p_{jk} y_j(t), \tag{26}$$

or in the vector form

$$y(t+1) = L\, y(t). \tag{27}$$

By applying Theorem 2 to (27), it follows that there exists a

$$z = \lim_{t \to \infty} \left[y(t) \Big/ \sum_{i=1}^{H} y_i(t) \right], \tag{28}$$

where z is given by (20). (28) is equivalent to

$$z_k = \lim_{t \to \infty} \left[\sum_{l=1}^{H} f(l,k;t) \Big/ \sum_{i=1}^{H} \sum_{j=1}^{H} f(j,i;t) \right], \qquad k \in N. \tag{29}$$

Since all $f(i,j;t)$ are non-negative, z_{ij} defined by (7) must exist for $i, j \in N$, and (23) follows from (29). It follows from (24) that

$$(1 + a_j)p_{jk} = \frac{f(j,k;t+1) / \sum_{l=1}^{H} \sum_{m=1}^{H} f(l,m;t)}{\sum_{i=1}^{H} f(i,j;t) / \sum_{l=1}^{H} \sum_{m=1}^{H} f(l,m;t)}. \tag{30}$$

Since the left side of (30) is constant, the limit of the right side of (30) exists as t goes to infinity, and (22) follows immediately. The proof is complete.

A similar limiting property for Case 3 holds, and is stated in the following theorem.

Theorem 4. Let $f(0,j;t)$ be given by (5) and let G be an H by H matrix

$$G = \left\{ g_{ij} \triangleq p_{ij} + \sum_{k=1}^{H} m_{ik} p_{kj};\ i,j \in N \right\}' \tag{31}$$

where g_{ij} is the (j,i)th element of G. If G is aperiodic and irreducible, then there exist z_{ij} defined by (7) for $i, j \in N$, and these z_{ij} are given by

$$z_{jk} = \left[\sum_{i=1}^{H} m_{ij} z_i + z_j \right] p_{jk}, \tag{32}$$

where

$$z_j = \sum_{i=1}^{H} z_{ij}, \tag{33}$$

and where z is the solution to

$$\left[\lambda z = G z,\ z \geqq 0,\ \sum_{i=1}^{H} z_i = 1, \quad \lambda \geqq 0 \right]. \tag{34}$$

Proof. It follows from (2), (5) that

$$y_k(t+1) = \sum_{i=1}^{H} g_{ik} y_i(t), \tag{35}$$

where $y_i(t)$ and g_{ik} are defined by (25) and (31) respectively. In vector form (35) is written as

$$y(t+1) = G\, y(t). \tag{36}$$

Hereafter, by using an argument similar to that used in the proof of the previous theorem, the proof of assertions (32)–(34) is straightforward.

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