Some Programming Problems in Population Projection

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This paper, which concerns a population model relating the birth rate to a stable age distribution and a stable rate of population growth, gives a nonlinear programming formulation based on this model that allows us to determine (1) the birth rate minimizing the sum of the costs due to changing the current birth rates and the costs of social goods and services associated with the resulting age distribution, and (2) the birth rate achieving a desired age distribution and population growth rate while minimizing the cost due to birth control.

ESTIMATING the changes in numbers of people of a population over time has been a major topic of research in population mathematics (KEYFITZ,[13,14] LESLIE,[15,16] POLLARD[24]). From a practical point of view, estimating the trajectory of a population can be required in various kinds of planning problems. For example, cities need estimates of the numbers of children who will be in different age groups over periods of time in order to decide how many schools to build. The same information is useful to a government in formulating its budget for family allowances, where allowances of this type exist.

There are several ways of stratifying a population for which a future projection is needed. The major factors by which a population is stratified are age and sex. Let us consider a female population stratified into several age groups. If one assumes that the rates of birth and death are constant, and that there is no immigration, he can show that the expected numbers of females classified by age groups satisfy a system of linear, first-order, homogeneous difference equations (Keyfitz,[13,14] Pollard[24]). Furthermore, if the individuals alive in the post-reproductive age groups are ignored, there is a stable age distribution covering only fertile ages (Keyfitz,[13,14] NAKAMURA,[18] Pollard[24]). In this paper we shall be concerned with the stable age distribution of a population of females as a function of birth rates.

Sometimes it becomes necessary to stratify a population by factors other than age and sex. For instance, a projection may be required that is stratified by income, race, marital status, and level of education, as well as age and sex. In this case, the method of linear difference equations would be impractical, since the coefficient matrix would probably be too large to handle (*see* ORCUTT, ET AL.,[21] p. 290). The simulation method suggested by Orcutt, et al.,[21] represents one way of handling this class of large problems for which the method based on a system of linear difference equations breaks down.

This simulation method based on each individual's decisions, which in turn determine his transition probabilities from his current state (including, say, the number of members of his family) to another, may be utilized to describe the situation in population planning in which, as stated by REINKE,[27] "... the ultimate actions concerning family limitation are usually recognized to be private family decisions."

Although the rates of birth and death are assumed to be given constants in mathematical studies of population models (Keyfitz,[13,14] Pollard[24]), some authors have studied the functional relations between the birth and death rates and other socioeconomic variables. SHEPS AND PERRIN[28] assume the changes in birth rates to be a function of contraceptive effectiveness, and conclude that the use of more effective methods by a smaller fraction of a population would produce a greater decline in the birth rate than the use of less effective methods by a larger portion of the population. ADELMAN[1] presents regression models that describe the birth and death rates for each of seven age groups of females in about thirty countries. The explanatory variables of the regression model for the age-specific birth rates are per capita real income, the percentage of the labor force employed outside of agriculture, the level of education, and the number of inhabitants per square mile. The explanatory variables for the death-rate model are per capita income, the percentage of the labor force employed outside of agriculture, the percentage rate of growth of per capita real income, and the number of physicians per 10,000 inhabitants.

HERMALIN[12] studies the effects of changes in death rates on population growth and the age distribution in the United States, and one of his conclusions is that the birth rate has been the chief determinant of age structure in this century, and that, if this trend continues, the birth rate will become the sole determinant of population growth. ORTIZ AND PARKER[23] report a simulation study of the possible impacts of a change of health status and/or population pattern upon mortality, life span, and the quality of life for various levels of two parameters: age-specific birth rates and age-specific death rates, which have been further classified according to the disease causing death. Assuming that there are two possible levels of values for each parameter, this simulation study determines (1) the age-stratified distribution of the population over time; (2) the behavior over time of the number of deaths, by age group and by cause of death, with emphasis on the change in the percentage composition; (3) the behavior of specific death rates by age group and the cause of death, and (4) the birth rates and population growth rates over time. Data from Costa Rica are used as an example.

The following section of this paper presents a mathematical model in which a stable age distribution and a stable rate of population growth are related to the birth rate. Then two kinds of costs, one resulting from changing the birth rate and the other associated with the age structure, are introduced, and a nonlinear programming problem is formulated to minimize a combination of these two costs with respect to the birth rate subject to technological constraints. We also consider a case in which a desirable stable age distribution is given. A quadratic programming problem is formulated for this case that enables us to find the birth rate that will achieve the given stable age distribution while minimizing the cost due to changing the current birth rate. Other family-planning models that use a mathematical programming approach are found, for example, in CORREA AND BEASLEY[10] and COULD AND MAGAZINE.[11]

NONLINEAR PROGRAMMING PROBLEMS IN BIRTH CONTROL

LET US CONSIDER a population of females at discrete intervals of time $t=0, 1, 2, \cdots$, and break the population into A age groups $\{1, 2, \cdots, A\}$ corresponding to unit intervals of time. We define $y_i(t)$ to be the expected number of females in age group i at time t. The probability that a female from age group i will survive to enter age group $i+1$ after a unit interval of time is q_i, which is positive for $i=1, 2, \cdots, A-1$, and is equal to zero for $i=A$. [Values of q_i for $i=1, 2, \cdots, A$ can be obtained from life tables.] Note that $1-q_i$ is the probability of death for a female in age group i during the unit time interval $(t, t+1)$. We further define b_i to be the probability that a female in age group i will give birth during the unit time interval $(t, t+1)$ to a single daughter and that this daughter is alive in age group 1 at time $t+1$ (*see* Note 1). Assuming that changes in the male population structure are consistent with the assumption of constant birth rates b_i and that births are independent of deaths, we derive the following system of linear difference equations

$$\begin{bmatrix} y_1(t+1) \\ y_2(t+1) \\ y_3(t+1) \\ \vdots \\ y_A(t+1) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & & b_{A-1} & b_A \\ q_1 & & & & \\ & q_2 & \ddots & & \\ & & & q_{A-1} & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ \\ y_A(t) \end{bmatrix}, \tag{1}$$

or, in a vector form,

$$y(t+1) = Ly(t), \tag{2}$$

where L is the coefficient matrix on the right side of (1). Since individuals in the post-reproductive age groups cannot affect the numbers in the reproductive age groups, we focus our attention on the reproductive age groups only, and assume that $b_A > 0$. The following two comments are in order.

1. The case of multiple births may be handled within the present framework by defining b_i to be $b_i = \sum_{j=1}^{j=N} jb_{ij}$, where b_{ij} is the probability of a female in age group i contributing exactly j daughters to age group 1 during the unit time period.

2. The case of immigration may be handled by adding an immigration-effect term to the right side of (2).

The following nonlinear-programming formulation holds with a slight modification for the immigration models considered by Pollard.[24]

We define the stable age distribution $z_1, z_2, \cdots, z_A$ by

$$z_i = \lim_{t\to\infty} y_i(t) / \textstyle\sum_{j=1}^{j=A} y_j(t), \tag{3}$$

where the $y_i(t)$ are the solution to (2). We also define the stable rate of population growth $\lambda-1$ by defining λ to be

$$\lambda = \lim_{t\to\infty} [\textstyle\sum_{i=1}^{i=A} y_i(t+1) / \sum_{i=1}^{i=A} y_i(t)]. \tag{4}$$

Nakamura[18] shows (*see* Note 2) that $z_1, z_2, \cdots, z_A$ and λ satisfy

$$\begin{aligned} \lambda z_1 &= b_1 z_1 + b_2 z_2 + \cdots + b_{A-1} z_{A-1} + b_A z_A, \\ \lambda z_2 &= q_1 z_1, \\ \lambda z_3 &= q_2 z_2, \\ &\vdots \\ \lambda z_A &= q_{A-1} z_{A-1}, \end{aligned} \tag{5}$$

where

$$z_1 + z_2 + \cdots + z_A = 1, \tag{6}$$

and

$$z_1, z_2, \cdots, z_A, \lambda \geqq 0. \tag{7}$$

It follows from (5) and (6) that

$$\lambda-1=(q_1+b_1-1)z_1+(q_2+b_2-1)z_2+\cdots+(q_{A-1}+b_{A-1}-1)z_{A-1}+(b_A-1)z_A. \quad (8)$$

This equation states that, if q_i+b_i-1 $(i=1, 2, \cdots, A-1)$ is negative (positive), age group i contributes in a negative (positive) way to the rate of total population growth $\lambda-1$. A similar comment holds for b_A-1. The so-called zero population growth rate (NEW YORK TIMES[19]) may be achieved if $q_i+b_i-1=0$ for $i=1, 2, \cdots$ $A-1$ and $b_A-1=0$. Assuming that λ is positive, we have from (5) and (6) that

$$z_1=(1+q_1/\lambda+q_1q_2/\lambda^2+\cdots+q_1q_2\cdots q_{A-1}/\lambda^{A-1})^{-1},$$
$$z_{i+1}=(q_i/\lambda)z_i. \qquad (i=1, 2, \cdots, A-1) \quad (9)$$

Let us assume that, to some degree, we can change the birth rates $b_1, b_2, \cdots,$ b_{A-1} and b_A subject to technological constraints

$$l_i \leqq b_i \leqq m_i, \qquad (i=1, 2, \cdots, A) \quad (10)$$

where l_i and m_i are given nonnegative constants.

Interest today seems to center mainly on decreasing birth rates. Hence, we will assume in this paper that

$$m_i=b_i(0). \qquad (i=1, 2, \cdots, A) \quad (11)$$

Let us further assume (*see* Note 3) that $w_i[b_i-b_i(0)]^2$ is an index of the cost per person in age group i of changing the present birth rate $b_i(0)$ of age group i to a new level b_i, where w_i is the weight assigned to age group i. Then the index of the cost of changing the present birth rate $b_i(0)$ to a new level b_i for all age groups is given by

$$\sum_{i=1}^{i=A} w_i[b_i-b_i(0)]^2. \quad (12)$$

Since changes in birth rates result in changes in the age distribution $z_1, z_2, \cdots,$ z_A, it would be helpful if we could assign a weight k_i to an average individual of age group i representing the cost paid for social goods and services for the average individual in this age group. Assuming that values can be arrived at for $k_1, k_2, \cdots,$ k_A, the sum of the average person's costs for each age group is represented by the index (*see* Note 4)

$$\sum_{i=1}^{i=A} k_i z_i. \quad (13)$$

One nonlinear birth control programming problem is:

PROBLEM I. *Minimize*

$$\sum_{i=1}^{i=A} w_i[b_i-b_i(0)]^2+\sum_{i=1}^{i=A} k_i z_i \quad (14)$$

with respect to b_1 $b_2, \cdots, b_A$ *subject to the constraints* (8), (9), *and* (10).

Once we have derived the optimal birth rates, the corresponding stable rate of population growth $\lambda-1$ is calculated from (8). While Problem I tries to balance the cost of birth control against the cost of social goods and services associated with a particular age distribution in choosing the optimal birth rates, it would be more useful in certain situations to find birth rates that guarantee a prescribed age distribution while minimizing the cost of birth control. Suppose $z_1^0, z_2^0, \cdots, z_{A-1}^0$ and z_A^0 constitute a given desirable age distribution satisfying (6) and (7). Suppose further that a specified population growth rate λ^0-1 is desired. Assuming that both the desired age distribution and population growth rate can be achieved by changing the present birth rates, we can formulate this problem as the following quadratic programming problem:

PROBLEM II. *Minimize* (12) *with respect to the constraints* (10) *and*

$$\lambda^0-1=(q_1+b_1-1)z_1{}^0+(q_2+b_2-1)z_2{}^0+\cdots+(q_{A-1}+b_{A-1}-1)z_{A-1}^0+(b_A-1)z_A{}^0. \quad (15)$$

Since the constraints (10) and (15) in Problem II are linear and very simple, one may overcome a possibly poor choice of λ^0, $z_1{}^0$, $z_2{}^0$, $\cdots$, z_{A-1}^0 and $z_A{}^0$ by solving the problem several times. That is, if the solution to Problem II does not exist for a particular choice of λ^0, $z_1{}^0$, $z_2{}^0$, $\cdots$, z_{A-1}^0, and $z_A{}^0$, we should revise the values of these parameters (say to λ^1, $z_1{}^1$, $z_2{}^1$, $\cdots$, z_{A-1}^1, and $z_A{}^1$) and try to solve Problem II again.

It can be shown that the solutions to Problem I and Problem II are optimal for two wider classes of nonlinear birth-control programming problems. An algorithm based on the linear-approximation method has been developed for Problem I, and Problem II may be solved by available quadratic programming algorithms (*see*, for example, ZANGWILL[35] for a general treatment of nonlinear-programming algorithms). Since the numerical values taken by the w_i's and the k_i's play an important role in determining optimal birth rates in our programming problems, we will discuss ways of choosing these cost parameters in the following section.

A DISCUSSION ON INDICES OF COSTS OF SOCIAL GOODS AND SERVICES

WE ARE CONCERNED with the costs of social goods and services and with how this burden is shared by taxpayers. Economists have traditionally been interested in how social goods and services and the resulting tax burdens are shared by various income groups, but, for a variety of sound reasons, little interest has been shown in the distribution of goods and services or the tax burden by age-group classifications (*see*, for example, MUSGRAVE,[17] pp. 135–162). It is, of course, true that income level has a strong positive correlation with age, if we consider people between, say, age 20 and age 50. It is also true that the redistributive effects of any given combination of social goods and services and taxes have much to do with the age distribution. Thus, in what follows we focus our attention on birth-control costs and other costs of social goods and services classified by age.

The demand for social goods and services such as welfare, education, health, police (crime prevention, traffic control, etc.), and recreation clearly is affected by the age structure of a population, although the demand for sanitation and fire prevention, for instance, may not be. Since our major interest is birth control and its impact upon society, we separate the cost due to birth control from costs of other social goods and services.

In order to find appropriate numerical values for the w_i's in expressions (12) and (14), we separate the amount of taxes (say, X_i) spent on birth control for age group i from the total amount of taxes (say, $\bar{X}_i$) spent for other social goods and services for age group i (*see* Note 5). [We do not distinguish between the female and male populations in this section.] Hence, $X_i+\bar{X}_i$ denotes the total amount spent for social goods and services for age group i. Let us assume that a birth-control program or a mix of birth-control programs is given, so that we know for each age group how to proceed if we wish to decrease the birth rate $b_i(0)$ by $\Delta b_i = b_i(0) - b_i$, and that the associated costs per person in age group i are represented by $w_i[b_i - b_i(0)]^2$, where b_i is some desirable birth rate such that $l_i \leqq b_i \leqq m_i = b_i(0)$. Assuming that the decrease in the birth rate Δb_i of age group i resulting from the prescribed birth-control program is known or can be estimated, we may take the weights entering into the index of birth-control costs per person in age group i to be $X_i/(\Delta b_i)^2 N_i$, where N_i is the number of people in age group i and where, as stated above, X_i is the amount of taxes spent on birth control for age group i. All quantities are measured with respect to a unit time period.

This procedure of choosing the w_i's assumes that from past data or experimental research it is possible (i) to estimate the X_i's and Δb_i's associated with the proposed birth-control program, even though such a program may not have been tried before, and (ii) to approximate X_i by a quadratic form of Δb_i, as is assumed in (12) and (14). (*See* Orcutt and Orcutt[22] for a description of how experimentation may be carried out at reasonable costs and in a socially acceptable manner.)

Since k_i should represent the tax burden per person in age group i resulting from expenditures for social goods and services other than birth control, we divide $\bar{X}_i$ (= total amount of local, state, and federal taxes spent for social goods and services other than birth control for age group i) first by the number of people in age group i to derive the total amount of local, state, and federal taxes spent for social goods and services other than birth control for an average person in age group i, and then, by the number of taxable people in the population to get k_i, the amount of tax burden borne by a taxable individual for social goods and services other than birth control for an average person in age group i. If we multiply k_i by z_i and sum $k_i z_i$ over i $(1 \leqq i \leqq A)$ we derive expression (13), our index of the costs, or average individual tax burden, from social goods and services weighted by age structure.

An important assumption underlying this procedure of choosing the k_i's is that k_i will be constant over time. Certainly there is no guarantee that this assumption holds in reality. In fact, one could argue that, in reality, k_i depends on the age structure $z_1, z_2, \cdots, z_A$ itself. In this respect it seems that Problem II is more appropriate than Problem I as a birth-control planning model, and that interest might reasonably center on the pattern of solutions to this model resulting from changing the model's parameters over what policy makers regard as the feasible ranges for these parameters.

A NUMERICAL STUDY

THIS SECTION PRESENTS the solutions to Problem II corresponding to several sets of hypothetically assigned cost data w_i. We define the unit interval of time to be five years. The population used in this example is that of females under 49 years of age in Costa Rica for 1963 (*see* Ortiz and Parker[23]), and is divided into ten ($A = 10$) age groups. The survival rates q_i, and the current birth rates of daughters per female in age group i, $b_i(0)$, are given in Table I. In this hypothetical study we assume that a linear combination of the Norwegian female age distribution for 1967 and the current (1963) female age distribution of Costa Rica is the desirable age distribution $z_1^0, z_2^0, \cdots, z_{10}^0$. (This assumption does not imply by any means that

TABLE I

Current Birth Rates, Lower Bounds, and Survival Rates

Age group i	1 (0–4)	2 (5–9)	3 (10–14)	4 (15–19)	5 (20–24)	6 (25–29)	7 (30–34)	8 (35–39)	9 (40–44)	10 (45–49)
$b_i(0)$ (= m_i)	0	0.000887	0.139018	0.529679	0.811426	0.762214	0.60319	0.383312	0.14211	0.01944
l_i	0	0	0.00007	0.09753	0.434347	0.42114	0.265035	0.13744	0.040869	0.0318
q_i	0.978551	0.995479	0.996319	0.994991	0.992594	0.990172	0.988557	0.984707	0.977851	—

Costa Rica should try to attain the Norwegian population pattern; this example is for expository purposes only—*see* Note 6.) We also used the Norwegian birth rates as lower bounds l_i. The age distribution of the Norwegian female population for 1967 was calculated from Table 6 in the UN Demographic Yearbook[31] and the lower bounds l_i, for the birth rates were derived (*see* Note 7) from Table 6 and Table 14 in the same source,[31] where the birth rates of daughters per female for age group i between 1963 and 1967 were summed to give l_i. The lower bounds l_i are given in Table I (*see* Note 8).

The desired age distribution $z_1^0, z_2^0, \cdots, z_{10}^0$ that we used in this study is defined as

$$z_i^0 = 0.8 z_i(CR) + 0.2 z_i(N) \tag{16}$$

for $i=1, 2, \cdots, 10$, where $z_i(CR)$ and $z_i(N)$ are the age distributions of females under age 49 in Costa Rica (1963) and in Norway (1967), respectively (*see* UN[31] and Arriaga,[2] p. 97). Now, $z_i(CR)$ and $z_i(N)$ are given in Table II, and the z_i and the $z_i(CR)$ are depicted in Fig. 1. The weights chosen in (16) imply a fairly small desired change in the age structure of the Costa Rican female population. Such a change is of the magnitude that we feel might be reasonable in a practical planning situation.

We chose three rates of stable population growth as possible desirable rates: (I) an annual rate of 3.5 per cent (18.7 per cent for five years), (II) an annual rate of 2.5 per cent (13.1 per cent for five years), and (III) an annual rate of 1.5 per cent (7.7 per cent for five years). The stable growth rate for the Costa Rican female population corresponding to the existing birth rates in 1963 is calculated from equation (8), and is shown to be an annual rate of increase of 4.1 per cent (22.1 per cent for five years). Finally the values (*see* Note 9) assigned to the w_i's are shown in Fig. 2. (Note that the absolute values of the w_i's in Problem II need not be specified and that only relative values are required.)

TABLE II

Age Distributions of Female Populations in Costa Rica (1963) and Norway (1967), and a Desired Age Distribution

Age group i	1	2	3	4	5	6	7	8	9	10
$z_i(CR)$	0.21816	0.17575	0.13867	0.10860	0.08698	0.07152	0.06344	0.05627	0.04382	0.03692
$z_i(N)$	0.11986	0.11448	0.11529	0.11336	0.11013	0.08101	0.07314	0.08117	0.09202	0.09955
z_i^0	0.19850	0.163496	0.133994	0.109552	0.09161	0.073478	0.06538	0.06125	0.05346	0.049446

A FORTRAN IV program written by RAVINDRAN[25,26] was used to solve Problem II for all combinations of the three specified desirable stable growth rates, three sets of cost data, and the desired age distribution. The results are given in Table III and Figs. 3–5.

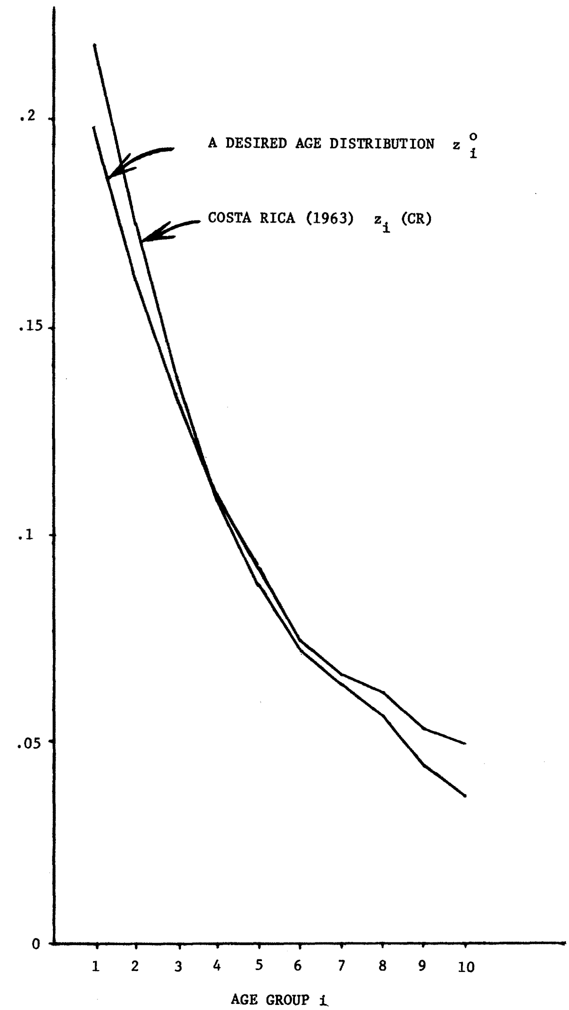

Fig. 1. The age distribution of the Costa Rican female population (1963) and the desired age distribution z_i^0.

Our computational results show that the optimal birth rates for most age groups corresponding to stable growth rate II are not higher than those corresponding to stable growth rate I, and similarly that the optimal birth rates for most age groups corresponding to stable growth rate III are not higher than those corresponding to stable growth rate II (*see* Figs. 3 and 4). While the optimal birth rates for different cost data differ somewhat from one another, no systematic differences were observed (Fig. 5).

Although we cannot generalize these observations from this particular numerical example, it does demonstrate the effectiveness of our nonlinear programming ap-

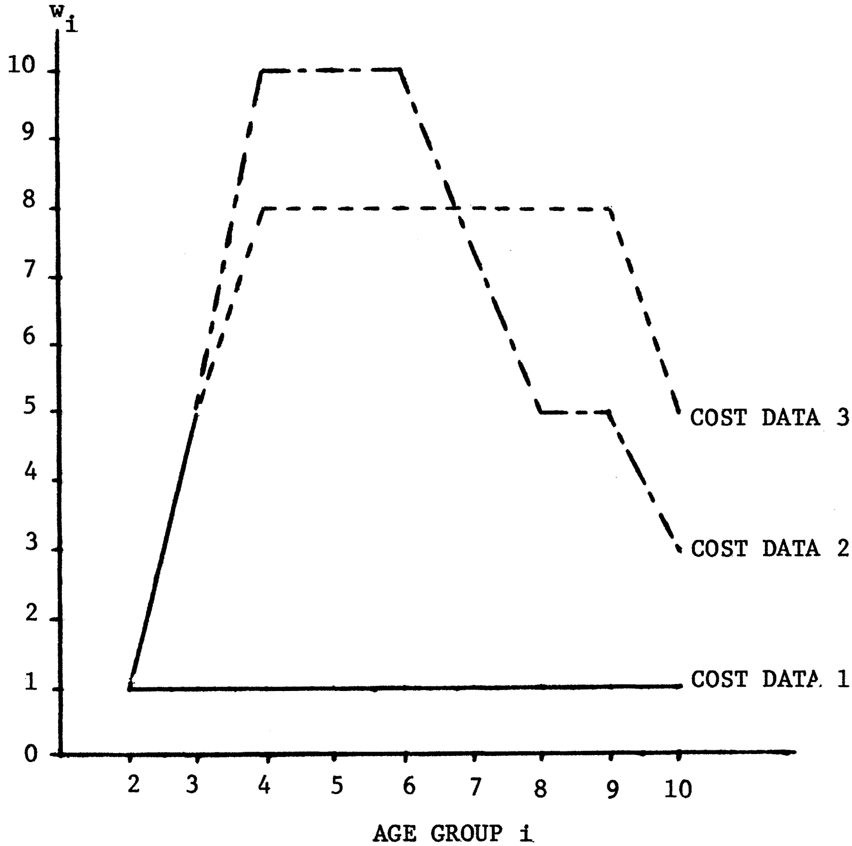

Fig. 2. Specification of the w_i's for cost data 1, 2, and 3.

proach in deriving some information that may not be attainable otherwise about the birth-control cost structure as related to an age structure.

CONCLUSION

THIS PAPER HAS presented a population model that relates the birth rate to a stable age distribution and a stable growth rate of population, and has formulated two nonlinear programming problems to derive the birth rate that optimizes specific cost criteria. As mentioned at the beginning of this paper, the birth rate is considered to be a function of many socioeconomic variables. Thus, ways of achieving

TABLE III

OPTIMAL BIRTH RATES OF DAUGHTERS PER FEMALE FOR FIVE YEARS

Growth rate	Cost data	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
	1	0.00000	0.06493	0.46910	0.76077	0.72162	0.56704	0.34944	0.11255	0.00319
I	2	0.00000	0.04550	0.49145	0.77946	0.73659	0.57467	0.34056	0.10480	0.00318
	3	0.00000	0.04548	0.48188	0.77145	0.73018	0.57466	0.35659	0.11878	0.00318
	1	0.00000	0.00007	0.32679	0.64177	0.62625	0.48211	0.26988	0.04310	0.00319
II	2	0.00000	0.00024	0.35706	0.66708	0.64653	0.47442	0.19029	0.04103	0.00318
	3	0.00000	0.00024	0.32962	0.64413	0.62814	0.48380	0.27146	0.04448	0.00318
	1	0.00000	0.00045	0.15764	0.55032	0.51289	0.38116	0.17531	0.04138	0.00318
III	2	0.00000	0.00024	0.17618	0.51582	0.52531	0.33948	0.13745	0.04103	0.00318
	3	0.00000	0.00008	0.15261	0.49611	0.50952	0.37817	0.17251	0.04104	0.00319

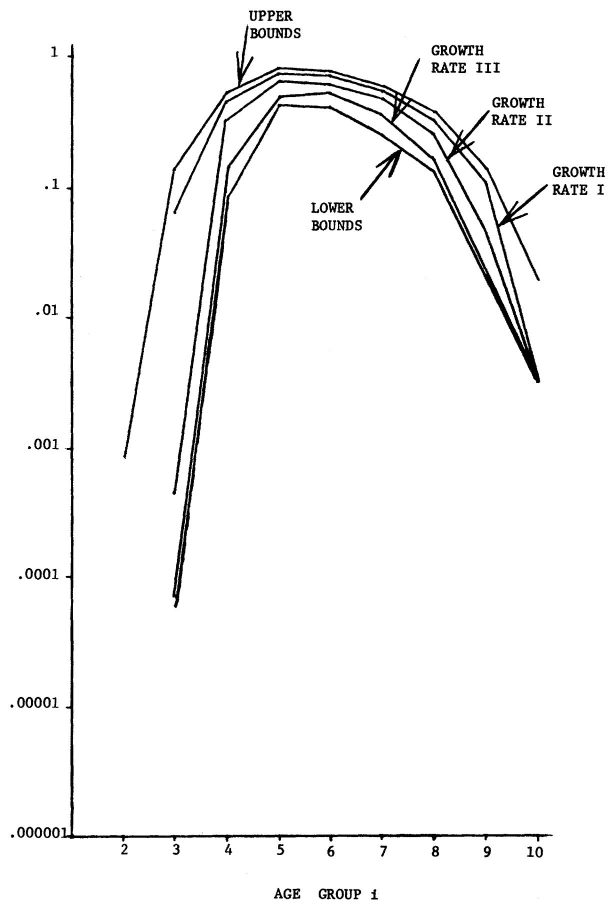

Fig. 3. The optimal birth rates for cost data 1 and growth rates I, II, and III.

the optimal birth rate (if such a rate exists in reality) derived from our nonlinear-programming formulation may be studied using an econometric approach of the type found in Adelman.[1] Such work may tell us which changes in which socio-economic variables or policies would result in the desired changes in birth rates.

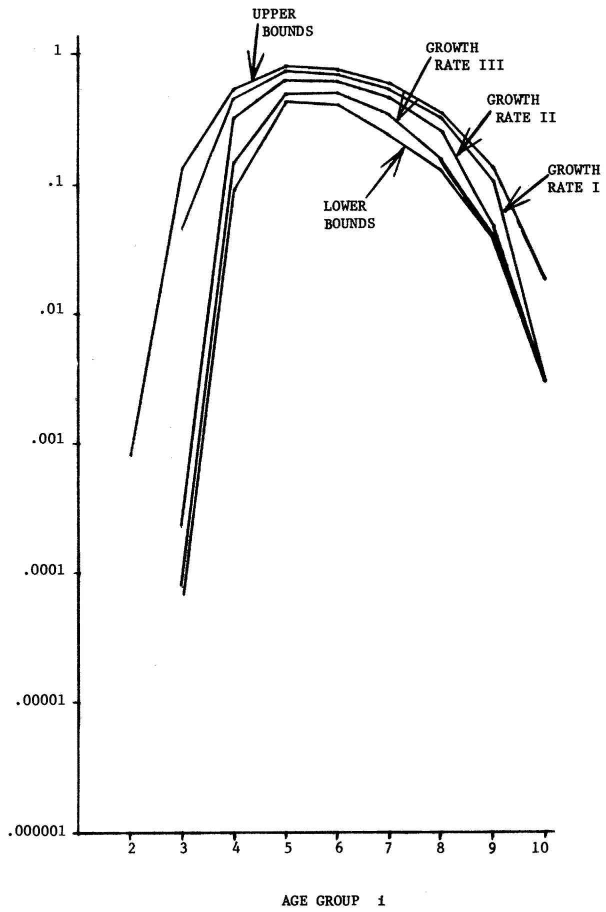

Fig. 4. The optimal birth rates for cost data 3 and growth rates I, II, and III.

NOTES

1. In this paper the birth rate b_i is defined to be the probability that a woman in age group i has a birth that survives to the end of the projection period, rather than the probability of a woman in age group i having a birth. In other words, the value b_i is the proportion of $y_1(t+1)$ who were born to women in $y_i(t)$. This is the method used to estimate b_i's in the numerical study of this paper.

2. The only assumption needed is that $q_i > 0$ for $i = 1, 2, \cdots, A-1$ and $b_A > 0$.

3. The quadratic birth-control cost index assumed here is by no means general, although it may describe the situation suggested by the study of CHOW AND LIU,[7] where, for the same birth-control program, the marginal cost was found to be increasing rather than constant. It is also true that the present formulation cannot take into account the fact that, in reality,

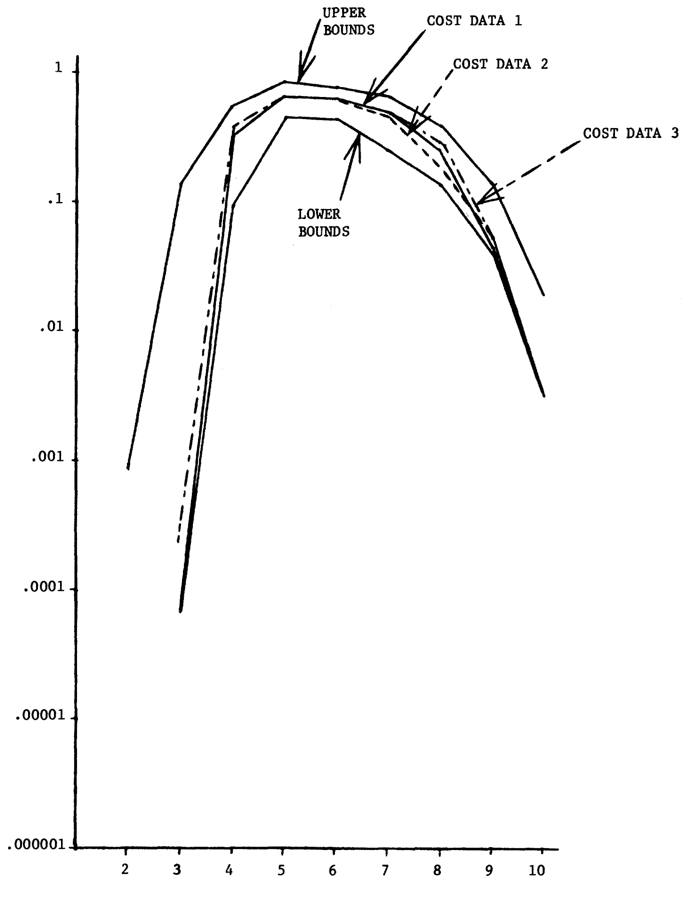

Fig. 5. The optimal birth rates for cost data 1, 2, and 3 and growth rate II.

different birth-control programs can lead to different costs over time with varying degrees of effectiveness. If the cost resulting from increasing the birth rate [i.e., $m_i > b_i(0)$] is also to be considered, an index of the birth-control cost of age group i may be given, in general by $w_i^1[b_i, b_i(0)]$ if $b_i < b_i(0)$, and by $w_i^2[b_i, b_i(0)]$ if $b_i \geq b_i(0)$, where $w_i^1[b_i, b_i(0)]$ and $w_i^2[b_i, b_i(0)]$ are some nonnegative functions of b_i and $b_i(0)$, respectively. We also note that the functional form of the birth-control cost is at least partially determined by how one chooses the parameters (such as the w_i's) involved in the cost expression.

4. The linear function form for the index of the costs of social goods and services given by (13) is by no means general (*see* Note 3). Depending on the estimation procedure specified for the k_i's one may find a more convenient or more suitable form.

5. Since expenditures for social goods and services are not in general classified by age groups, we may have to estimate $\bar{X}_i$ from the utilization rate of social goods and services by age group i. In this case, we must first estimate the amount of each category of goods and services used by each age group in a unit period of time and then use information and data on the costs of providing each category of goods and services to approximate $\bar{X}_i$.

6. The difficulties of reducing birth rates in Latin American countries to duplicate the European population pattern are examined in Arriaga,[3] pp. 193–211. The author concludes that: "Various ways to obtain the decline of fertility necessary to give Latin American countries the same crude birth rate as Europe at the same mortality level have been analyzed with the conclusion that the European pattern would not have been possible in Latin America."

7. Since a breakdown by sex of newborn babies was not available for 1964 and 1967, it was assumed that the ratios of female babies to the total number of newborns in these years are the same as that of 1965. It was also assumed that all mothers under age 14 who gave births to daughters belonged to the age group between 10 and 14. Thus, we have $l_1 = l_2 = 0$.

8. Studies on changing birth rates and the impacts of these changing birth rates on populations in Latin American countries are found, for instance, in Arriaga,[3] Collver,[9] and Chaplin.[6] Similar studies for the US population are found in Whelpton,[32] Whelpton, et al.,[33] Thompson and Whelpton,[30] Orcutt, et al.,[21] Yasuba,[34] Coale and Zelnik,[8] and Okun.[20]

9. Although many authors have studied the costs of reducing the birth rates of different age groups (*see*, for example, Berchman, et al.,[4] Berelson,[5] and Sheps and Ridley[29]), it is not yet clear exactly how the costs differ from one age group to another, since they seem to depend significantly on the parity as well as age of a mother. The values assigned to the w_i's in this numerical study should not be considered to represent the actual situation in Costa Rica.

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