POLI574 – Discrete Choice

Acknowleding a great debt to Matt Golder's notes, themselves dependent on Train (2007)

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Modeling Categorical Outcomes

- Dependent variable is unordered categories
  - Vote choice
  - Choice of policy instrument
  - Outcome of inter-state interactions (e.g. war, trade)
- OLS doesn't work, except LPM for 2 categories
- Logit/Probit are also for 2 categories
- Frequently two outcomes 'closer' together than to other outcomes (see 'IIA' later)
- Frequently nested choices or selection effects

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But first… review Binary Dependent Variables

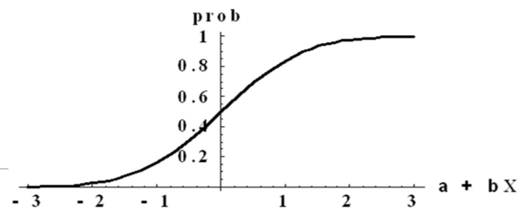
- Recall the linear probability model, which can be written as $P(y = 1|\boldsymbol{x}) = \beta_0 + \boldsymbol{x\beta}$
- An alternative is to model the probability as a function, $G(\beta_0 + \boldsymbol{x\beta})$, where $0<G(z)<1$
- This G just translates – or *squishes* -- the linear additive model into the 0 to 1 space

Logit

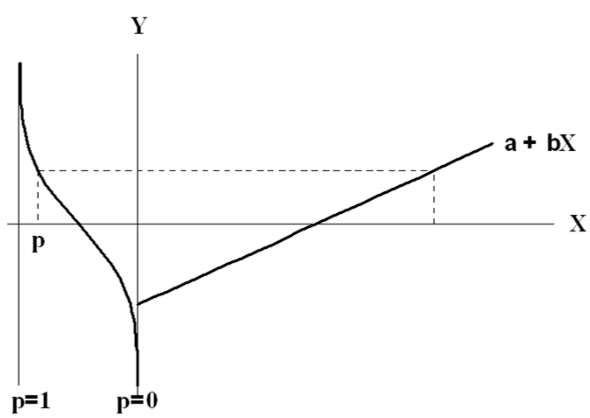
- A common choice for G(z) is the logistic function, which is the *cumulative distribution function* for a standard logistic random variable
- $G(x\beta) = \exp^{(x\beta)}/[1 + \exp^{(x\beta)}]$
  or $1/[1\text{-}\exp^{-x\beta}]$
- We're taking numbers from - ∞ to + ∞ and transforming those numbers using this cumulative distribution function

Binary Data – View 1 (CDF)

- View 1 – we compute a number that is a linear combination of our predictors, call it $y=\alpha+\beta x$. We then convert y into a probability p by using a cumulative distribution function (CDF).
  Our outcome is 1 with probability p.



Another CDF View



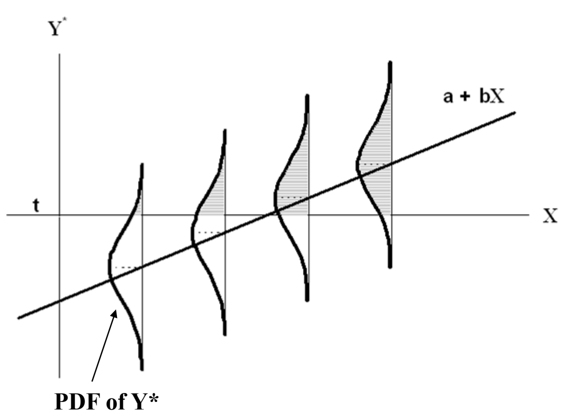
Binary Data – View 2 (Latent or Unobserved Variable)

- View 2 – we compute a number that is a linear combination of our predictors and then add an error term, call it

  $y^* = \alpha + \beta x + u$

  We then get an outcome of 1 if $y^* >= 0$, outcome 0 if $y^* < 0$. In this case, the probabilistic element is the error term u, and y^* is an <u>unobserved variable</u>.

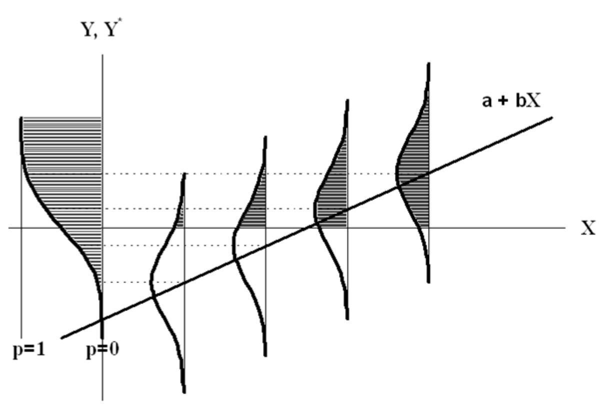
Binary Data – Unobserved Variable View



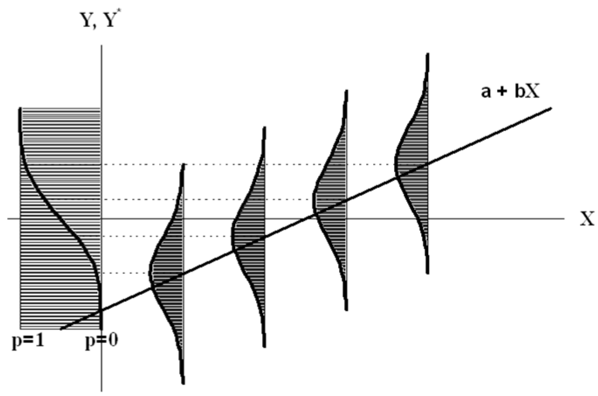
Comparing CDF and Latent Variable Views

- The two views are equivalent. Each one can be converted into the other, where the cumulative probability function (CDF) in view 1 matches the CDF of the distribution of *u* in view 2.

Combining the Two Views



Combining the Two Views



These are all NONLINEAR models

- The rate of change in the dependent var with respect to the independent var
  IS NOT CONSTANT
- So we have to estimate coefficients by
  trial and error
- So… maximum likelihood

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Likelihood and Traditional Probability

- Theory of likelihood is the reverse of traditional probability theory
- Traditional theory: probability that we got this set of data given the TRUE parameter values
- In likelihood we're honest that we only have one set of data. So we talk about the 'likelihood' of each set of parameter values given the data we actually got
- What model (i.e. parameters) is most likely to have produced the data we collected?

Likelihood is a RELATIVE measure of uncertainty

- The *likelihood function* is a measure of the relative probability of all possible parameter values (i.e. estimates of the true model)
  - think of all possible parameter values. Whoah!
- So it gives us a mean (most likely parameter value) and a variance (how much more likely than others)
- The maximum of this function gives us an estimate of the mean of the parameter (vector)
- THIS APPLIES TO ALL POSSIBLE MODELS

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Constructing a Likelihood (logit)

We assume a *data generating process*

- This applies to every observation
- For binary outcomes we assume they are generated by a Bernoulli distribution: $p_i^{y_i}(1-p_i)^{1-y_i}$
- Then we model *p*, the probability (our model), as a function of expalanatory variables: $p_i = g(x_i, \beta)$\
- For logit, let $p_i = \frac{1}{1+\exp(-x_i\beta)}$
- Now, since our observations are independent…
- The probability of all of the Y given one particular value of *p* (i.e. the model) is equal to the product of all the probabilities $\Pr(Y \mid p) = \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i}$
- So we combine these and get

$$\Pr(y \mid \beta) = \prod_{i=1}^{n}\left(\frac{1}{1+\exp(-x_i\beta)}\right)^{y_i}\left(1-\frac{1}{1+\exp(-x_i\beta)}\right)^{1-y_i}$$

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Constructing a Likelihood Continued

From $$\Pr(y \mid \beta) = \prod_{i=1}^{n}\left(\frac{1}{1+\exp(-x_i\beta)}\right)^{y_i}\left(1-\frac{1}{1+\exp(-x_i\beta)}\right)^{1-y_i}$$

The theory of maximum likelihood says that the likelihood *function* $L(\beta|y)$ is proportional to this expression

So to get the log-likelihood that's easier to work with, we take the log of the expression and we get

$$\ln L(\hat{\beta} \mid y) = \sum_{i=1}^{n} - y_i \ln\left[1+\exp(-x_i\hat{\beta})\right] - (1-y_i)\ln\left[1+\exp\left(-x_i\hat{\beta}\right)\right]$$

We've gone from products to sums and from wanting to minimize something to maximizing this function

We plug in values for β, call them $\hat{\beta}$, and do an astronomical amount of simple arithmetic to get a log-likelihood for that set of estimates.

Then we use an algorithm to search for the set of estimates that maximizes this log-likelihood

Now, Multiple Outcomes

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Notation (follows Golder)

- ***n*** individual cases (decision makers)
- ***J*** alternatives
- ***i and j*** are alternative outcomes
  - ***i*** chosen outcome (choice)
  - ***j*** all outcomes (alternatives)
- $\boldsymbol{\beta_j}$ is the set of coefficients for alternative ***j***
  (where one set is set to zero as the 'base category')
- ***X*** is still the linear-additive independent variables

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Random Utility Model

- Differences in utility of alternatives result in choice / behaviour
- But a random component, so we get a predicted behaviour given characteristics of choices and choosers
- Probability of each outcome for each chooser
- Or: Proportion of each choice within population groups defined by combinations of characteristics

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RUM

$$\begin{aligned} P_{ni} &= \text{Prob}(U_{ni} > U_{nj} \forall j \neq i) \\ &= \text{Prob}(V_{ni} + \epsilon_{ni} > V_{nj} + \epsilon_{nj} \forall j \neq i) \\ &= \text{Prob}(\epsilon_{nj} - \epsilon_{ni} < V_{ni} - V_{nj} \forall j \neq i) \end{aligned}$$

- This last step is weird
- It expresses the probability as:
  - *i* is chosen if the difference between the errors is less than the difference between the systematic difference in utilities
- Just like OLS in that the model minimizes the residuals – the ϵ
- Just like all MLE in that we choose a distribution for these errors
- Then to get probabilities we calculate the integral of these unobserved utilities
  - i.e. the probability that *i* is chosen is how much probability mass is below the threshold where the difference in the errors is more than the difference in the systematic portion of the utilities.

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Differences in Utility

- As Golder says: "Only Differences in Utility Matter"
- Because utility is *unobserved* or 'latent', and we only know whether one alternative was chosen as opposed to another, we can only think of systematic influences as *relative*
- So the impact of a characteristic of a chooser (e.g. female) **is not** that it produces, on average, Θ_{n1} and Θ_{n2} and so on *U*tilities for the choices.
- Instead, it just tells us about the average *difference* in the utility of the two choices, i.e. $\Theta_2 - \Theta_1$
- Since we don't observe utility, that $\Theta_2 - \Theta_1$ is indeterminate, so we just set one of them to ZERO and interpret the Θ_i parameter as the difference in the utility of the i^{th} choice from the one choice for which we set all the Θ's to zero.

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Logit Models for categorical outcomes

- Assume a distribution for the ϵ
- We actually use one that's mathematically convenient rather than substantively justified
  - Suffice to say it is a logistic dist. for choice btw any two alternatives
  - $\frac{e^{\tilde{\epsilon}_{nji}}}{1+e^{\tilde{\epsilon}_{nji}}},\ where \tilde{\epsilon} = \epsilon_{nj} - \epsilon_{ni}$
- BIG assumption is that the unobserved part of the utility of one alternative is **independent** of the unobserved part of other alternatives (IIA, more later)
- Means you've got a good, well-specificed model: one that includes all systematic influences on the choices

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Multiple Outcome Logit Choice Probabilities

- So the choice of one alternative by a chooser indicates that the error for each other choice was below
  $\epsilon_{ni} + V_{ni} - V_{nj}$
- With multiple choices, we need the probability that this is true for all $j \neq i$, which is the product of all of the cumulative distributions of the errors for all the non-chosen choices, relative to the distribution of the errors of *I*
  (that's roughly what Golder's eq. 16 says)
- That's the criterion analogous to 'least-squares' for OLS
- So the MNL choice probabilities are

$$P_{ni} = \frac{e^{x_{ni}\beta}}{\sum_j e^{x_{nj}\beta}}$$

- And the log likelihood is this over all choices and choosers

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Two models, MNL and CoLogit

- Golder does Conditional Logit before Multinomial Logit
- Weird choice, but it makes a bit of sense
- I'm going to follow him

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Conditional Logit

- Pure Conditional Logit involves only characteristics of choices
- Transportation models involved price, speed, comfort of each of modes of transport
- Notice that the *x* are subscripted by *nj*, meaning they are about the decision-maker *relative* to the alternatives

$$U_{nj} = V_{nj} + \epsilon_{nj}$$
$$= x_{nj}\beta + \epsilon_{nj}$$

- Like 'distance' from a party on policy, or a country's distance from potential allies or adversaries
- β **has no subscript** because the effect of this variable is constant across alternatives
  - E.g. 'distance' or higher price makes you less likely to choose something
  - Speed, comfort make choice more likely

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Conditional Logit in Stata

Vote Choice in Quebec, 2011

- `clogit choice samelang dist_corptax, group(id)`

```
Iteration 0:   log likelihood = -2197.0125
Iteration 1:   log likelihood = -2196.8142
Iteration 2:   log likelihood = -2196.8142

Conditional (fixed-effects) logistic regression   Number of obs   =       7428
                                                  LR chi2(2)      =      42.77
                                                  Prob > chi2     =     0.0000
Log likelihood = -2196.8142                       Pseudo R2       =     0.0096

------------------------------------------------------------------------------
      choice |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    samelang |   .5009395   .0762589     6.57   0.000     .3514749    .6504041
dist_corptax |  -.0933343   .0478339    -1.95   0.051    -.1870869    .0004184
------------------------------------------------------------------------------
```

- Coefficients are change in log-odds of choosing an alternative, for one-unit change in the independent variable

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Multinomial Logit (MNL)

$$U_{nj} = V_{nj} + \epsilon_{nj}$$

The systematic component of the utility function is given as:

$$V_{nj} = z_n \gamma_j$$

So, we have

$$U_{nj} = z_n \gamma_j + \epsilon_{nj}$$

- *z* is equivalent to *x* variables
- *γ (gamma)* is equivalent to β
- note that the *γ* are subscripted, so separate 'effects' of each *z* (characteristic) on each choice
  - E.g. **female** may have different effects on prob of choosing each party
  - Trade deficit may have a different effect on choice of trade war, unilateral tarriff reduction, bilateral negotiation, or multilateral trade negotiation
- MNL Choice Probabilities: $P_{ni} = \frac{e^{x_{ni}\beta}}{\sum_j e^{x_{nj}\beta}}$

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MNL identification

- Attributes of choosers don't vary across alternatives
- So they can only create differences between alternatives
  - e.g. educ level can only make some parties more likely to be voted for
- Simple solution: set all coefficients for one alternative to zero
- Coefficients are always about the difference in choice probabilities between two of the choices
- As a decision-maker becomes more likely to choose one alternative, she is less likely to choose others
- This just works out to a different set of independent variables. The likelihoods are basically the same.

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MNL is binary logits!

- MNL estimates the same parameters as a series of binary logits
- It's slightly more efficient (see Alvarez and Nagler)
- This is because of IIA
- Later, we'll talk about relaxing IIA

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Digression: Don't estimate choice versus all others

- … unless you have a theoretical reason to
- Cautionary tale:
  IS BQ voting influenced by attitude to spending on Envrmt?

```
. logit vote4 sov  spend_EN

Logistic regression                               Number of obs   =        904
                                                  LR chi2(2)      =     243.17
                                                  Prob > chi2     =     0.0000
Log likelihood =  -430.4643                       Pseudo R2       =     0.2202

------------------------------------------------------------------------------
       vote4 |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         sov |   2.455879    .179795    13.66   0.000     2.103487    2.808271
    spend_EN |   .1583196   .1679772     0.94   0.346    -.1709097    .4875489
       _cons |  -2.592755   .4599843    -5.64   0.000    -3.494307   -1.691202
------------------------------------------------------------------------------
```

- No effect of Environment attitudes?

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MNL in Stata

```
. mlogit vote sov  spend_EN

Multinomial logistic regression                   Number of obs   =        904
                                                  LR chi2(8)      =     353.81
                                                  Prob > chi2     =     0.0000
Log likelihood = -1149.7148                       Pseudo R2       =     0.1333
 ------------------------------------------------------------------------------
        vote |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
Liberal      |
         sov |  -3.323585   .2804922   -11.85   0.000    -3.873339    -2.77383
    spend_EN |  -.0180076   .2244897    -0.08   0.936    -.4579993    .4219842
       _cons |   1.033057   .6140084     1.68   0.092    -.1703772    2.236492
-------------+----------------------------------------------------------------
Conservati~s |
         sov |  -3.063571   .2648611   -11.57   0.000    -3.582689   -2.544453
    spend_EN |  -.9299179   .2053723    -4.53   0.000     -1.33244   -.5273956
       _cons |   3.380755   .5476791     6.17   0.000     2.307323    4.454186
-------------+----------------------------------------------------------------
NDP          |
         sov |  -1.929275   .2010957    -9.59   0.000    -2.323415   -1.535135
    spend_EN |   .0810441   .1944765     0.42   0.677    -.3001229    .4622111
       _cons |   .8961252   .5389201     1.66   0.096    -.1601387    1.952389
-------------+----------------------------------------------------------------
Bloc_Quebe~s |  (base outcome)
-------------+----------------------------------------------------------------
Green_Party  |
         sov |  -1.252255   .3907826    -3.20   0.001    -2.018175   -.4863351
    spend_EN |   1.313919   .6122771     2.15   0.032     .1138781     2.51396
       _cons |  -4.987046   1.783435    -2.80   0.005    -8.482515   -1.491578
------------------------------------------------------------------------------
```

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