Integral Calculus

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Absolute Minimum/Maximum Values (sec. 12.8)

Definition 1. Let f be defined in a set $\mathcal{D}$ in $\mathbb{R}^2$ containing the point (a,b). If $f(a,b) \geqslant f(x,y)$ for every (x,y) in $\mathcal{D}$, then $f(a,b)$ is an **absolute maximum value** of f on $\mathcal{D}$. If $f(a,b) \leqslant f(x,y)$ for every (x,y) in $\mathcal{D}$, then $f(a,b)$ is an **absolute minimum value** of f on $\mathcal{D}$

Difference between Absolute Min/Max and Local Min/Max

If (a,b) is a local maximum, then $f(a,b)$ is the largest value of $f(x,y)$ in an open disk around (a,b) **but** it is possible to find larger value of $f(x,y)$ in its domain. So, "**local maximum $\leqslant$ absolute maximum**"
If (a,b) is a local minimum, then $f(a,b)$ is the smallest value of $f(x,y)$ in an open disk around (a,b) **but** it is possible to find smaller value of $f(x,y)$ in its domain. So, "**local minimum $\geqslant$ absolute minimum**"

Finding Absolute Min/Max Values on closed bounded sets

Let f be continuous on a closed bounded set $\mathcal{R}$ in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on $\mathcal{R}$:

1. Determine the values of f at all critical points in $\mathcal{R}$.
2. Find the maximum and minimum values of f on the boundary of $\mathcal{R}$.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on $\mathcal{R}$, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on $\mathcal{R}$

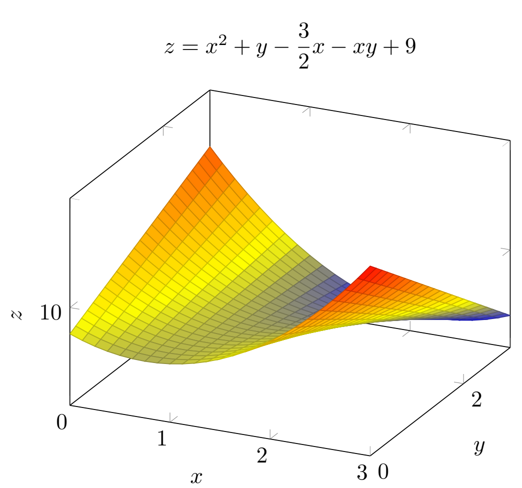
Example 0.1. Let $z = f(x,y) = x^2 + y - \frac{3}{2}x - xy + 9$ and $\mathcal{R}$ is the closed region bounded by the triangle with vertices (0,0), (2,0) and (0,2).
Step 1: Find the value of the critical points inside $\mathcal{R}$.

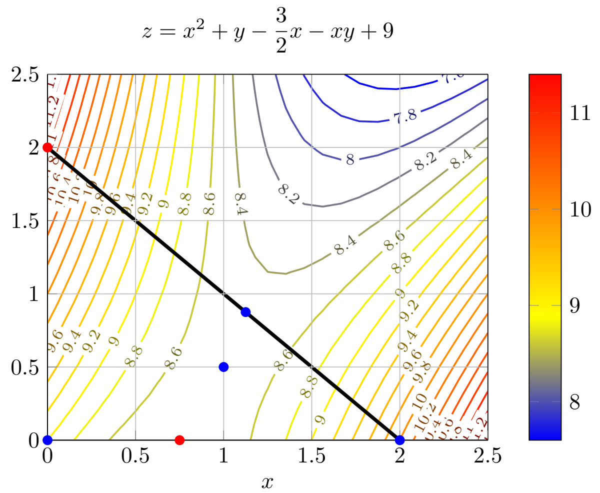
$$f_x = 2x - \frac{3}{2} - y \quad = 0 \tag{1}$$
$$f_y = 1 - x \quad = 0 \tag{2}$$

From Eq. (2), we have $x = 1$. Substituting into Eq. (1), we have $y = \frac{1}{2}$. The critical point is $(1, 1/2)$.

$$\boxed{f(1,\frac{1}{2}) = 1^2 + \frac{1}{2} - \frac{3}{2}(1) - 1(\frac{1}{2}) + 9 = \frac{17}{2}}$$

Step 2: Find the min and max on the boundary of $\mathcal{R}$. The boundary of $\mathcal{R}$ consists of vertices of the triangle C_1, C_2 and C_3. We will consider them separately:





- $C_1 = \{(x,y)|x=0 \text{ and } 0 \leqslant y \leqslant 2\}$.
  On C_1 we have $g_1(y) = f(0,y) = 0^2 + y - 0 - 0(y) + 9 = y + 9, \quad 0 \leqslant y \leqslant 2$. Then, we check for critical points and the value of f at the ending points of C_1:
  $$g_1'(y) = (y+9)' = 1 \neq 0 \Longrightarrow \text{no critical points}$$
  So at the ending points of C_1, we have $\boxed{g_1(0) = f(0,0) = 9 \text{ and } g_1(2) = f(0,2) = 11}$

- $C_2 = \{(x,y)|y=0 \text{ and } 0 \leqslant x \leqslant 2\}$.
  On C_2, we have $g_2(x) = f(x,0) = x^2 + 0 - \frac{3}{2}x - x(0) + 9 = x^2 - \frac{3}{2}x + 9, \quad 0 \leqslant x \leqslant 2$. Then, we check for critical points and the value of f at the ending points of C_2:
  $$g_2'(x) = 2x - \frac{3}{2} = 0 \Longrightarrow x = \frac{3}{4} \Longrightarrow \text{critical point } (\frac{3}{4}, 0)$$
  So $\boxed{g_2(\frac{3}{4}) = f(\frac{3}{4}, 0) = \frac{135}{16} = 8.4375}$

  And at the ending points of C_2, we have $\boxed{g_2(0) = f(0,0) = 9 \text{ and } g_2(2) = f(2,0) = 10}$

- $C_3 = \{(x,y)|y = -x+2 \text{ and } 0 \leqslant y \leqslant 2\}$.
  On C_3, we have $g_3(x) = f(x, -x+2) = x^2 + (-x+2) - \frac{3}{2}x - x(-x+2) + 9 = x^2 - \frac{9}{2}x + 11, \quad 0 \leqslant x \leqslant 2$.
  Then, we check for critical points and the value of f and the value of f at the ending points of C_3:
  $$g_3'(x) = 4x - \frac{9}{2} = 0 \Longrightarrow x = \frac{9}{8} \Longrightarrow \text{critical point } (\frac{9}{8}, \frac{7}{8})$$
  So $\boxed{g_3(\frac{9}{8}) = f(\frac{9}{8}, \frac{7}{8}) = 8.4687}$

  And at the ending points of C_3, we have $\boxed{g_3(0) = f(0,2) = 11 \text{ and } g_3(2) = f(2,0) = 10}$

Step 2: Comparison from steps 1 and 2
$$f(1, \frac{1}{2}) = 8.5; \quad f(2,0) = 10; \quad f(0,0) = 9; \quad f(0,2) = 11; \quad f(\frac{3}{4}, 0) = 8.4375; \quad f(\frac{9}{8}, \frac{7}{8}) = 8.4687$$

$\boxed{\text{So } f(0,2) = 11 \text{ is the max absolute and } f(\tfrac{3}{4}, 0) = 8.4375 \text{ is the min absolute}}$