Integral Calculus

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1 Integration by Parts

It is a useful method to evaluate integrals of products of functions. Given two differentiable functions u and v, the product rule states:

$$\frac{d}{dx}(u(x) \cdot v(x)) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

By integrating both sides, we can write this rule in terms of an indefinite integral:

$$u(x) \cdot v(x) = \int \frac{d}{dx}(u(x) \cdot v(x)) = \int u'(x) \cdot v(x)dx + \int u(x) \cdot v'(x)dx$$

Rearranging the expression, we have:

$$\int u(x) \cdot \underbrace{v'(x)dx}_{dv} = u(x) \cdot v(x) - \int v(x) \cdot \underbrace{u'(x)dx}_{du}$$

Here, the goal is to suppress the independent variable x. We can write it as:

Indefinite

$$\int u \; dv = uv - \int v \; du$$

Definite

$$\int_a^b u(x)\; v'(x)dx = u(x)\; v(x)\Big|_a^b - \int_a^b v(x)\; u'(x)dx$$

OR

$$\int_a^b u \; dv = uv - \int_a^b v \; du$$

Example 1.1. Integration by parts. Evaluate $\int xe^x dx$

Functions in original integral	$u = x$	$dv = e^x dx$
Functions in new integral	$du = dx$	$v = e^x$

$$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = \underbrace{x}_{u} \; \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{dx}_{du}$$

We have:

$$\int x\, e^x dx = xe^x - \int e^x dx = xe^x - e^x + c = (x-1)e^x + c$$

When is the "Integration by parts" the best choice to evaluate an integral?

1. The integrand is a product of functions $(u \cdot dv)$
2. u is easy to differentiate and v is easy to integrate

The integrand is a product of functions

Many books mention the ILATE rule of thumb here. These are supposed to be memory devices to help you choose your u and dv in an integration by parts question. We have:

- I = inverse trigonometric
- L = logarithmic
- A = algebraic
- T = trigonometric
- E = exponential

ILATE rule is supposed to suggest the order in which we choose the u.

Example 1.2.

$$\int \underbrace{x}_{A} \underbrace{\tan^{-1} x}_{I} dx \Longrightarrow u = \tan^{-1} x \qquad dv = xdx$$

$$\int \underbrace{x^2}_{A} \underbrace{\ln x}_{L} dx \Longrightarrow u = \ln x \qquad dv = x^2 dx$$

Example 1.3. Integration by parts. Evaluate the following integral:

$$\int_1^e x \ln x dx$$

Here, we have an algebraic and a logarithmic functions. He have:

$$\begin{array}{rcc} \text{Functions in original integral} & u = \ln x & dv = xdx \\ \text{Functions in new integral} & du = \dfrac{1}{x} dx & v = \dfrac{x^2}{2} \end{array}$$

$$\int_1^e x \ln x dx = \ln x \left. \frac{x^2}{2} \right|_1^e - \int_1^e \frac{x^2}{2} \frac{1}{x} dx = \ln x \left. \frac{x^2}{2} \right|_1^e - \int_1^e \frac{x}{2} dx$$

$$\int_1^e x \ln x dx = \ln(e) \frac{e^2}{2} - \ln(1) \frac{1^2}{2} - \left. \frac{1}{2} \frac{x^2}{2} \right|_1^e = \ln(e) \frac{e^2}{2} - \frac{1}{4}(e^2 - 1)$$

$$\int_1^e x \ln x dx = \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}$$

Related Exercises sec. 7.2 7–22

Example 1.4. Repeated use of integration by parts. Evaluate the following integral:

$$\int x^2 e^x dx$$

From the ILATE rule, we can choose $u = x^2$ and $dv = e^x dx$ Hence, we have:

Functions in original integral	$u = x^2$	$dv = e^x dx$
Functions in new integral	$du = 2x dx$	$v = e^x$

$$\int x^2 e^x dx = x^2\ e^x - \int e^x\ 2x\ dx = x^2\ e^x - 2\int xe^x\ dx$$

Here we have to **apply again the Integration by Parts** on $\int xe^x\ dx$. Therefore, we choose again u and dv.

Functions in original integral	$u = x$	$dv = e^x dx$
Functions in new integral	$du = dx$	$v = e^x$

Finally, we have:

$$\int x^2 e^x dx = x^2\ e^x - 2\left(xe^x - \int e^x dx\right) = x^2 e^x - 2[(x-1)e^x] + c$$

So for $\int x^n e^x\ dx$, where n is a positive integer, we apply the IBP $n-$times. Hence,

$$\int x^n e^x\ dx = x^n e^x - n\int x^{n-1} e^x\ dx$$