

Integral Calculus

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10. Numerical Integration

The reason why we spent so much time on antiderivatives was so that we could find and evaluate the integral $\int_a^b f(x) dx$! But, sometimes an integrand may not have an obvious (nice) antiderivative, such as e^{x^2} , $\cos(x^2)$, $\sqrt{1+x^3}$, $\sin(x^2)$ and $\frac{1}{\ln x}$. To deal with these types of integrands (functions), we need to approximate the integral. For these functions the "analytical methods" fail, so we turn to "numerical methods", which are typically done on a calculator or computer.

Analytical Methods \rightsquigarrow "Exact value"

Numerical Methods \rightsquigarrow "Approximate Value"

Note:

When "Analytical Methods" fail to evaluate $\int_a^b f(x) dx$, we use "Numerical Methods" to approximate the integral $\int_a^b f(x) dx !!!$

In this section, we introduce three (3) rules (Midpoint, Trapezoid and Simpson's Rules) to approximate the integral of the form $\int_a^b f(x) dx$. We start by the definition of "Absolute and Relative Error".

In this approximations, it is important for us to get the better estimate. In other words, we want to minimize our Error. but how can we define the error?

To find the error, we need to compare the approximate value with the exact value. Then, the minimum value for the error means we have the better approximation.

10.1 Absolute and Relative Error

Because numerical methods do not typically produce exact results, we should be concerned about the accuracy of approximations, which leads to the ideas of "absolute" and "relative error".

Definition (Absolute and Relative Error): approximate value
exact value

Suppose c is approximation (computed numerical solution)

x is exact solution (exact value)

Then,

$$\text{absolute error} = |c - x|$$

$$\text{relative error} = \frac{|c - x|}{|x|} \quad (\text{if } x \neq 0)$$

Example 1. Compute the absolute and relative errors in using 1.414 to approximate $\sqrt{2}$.

Solution: we are approximating $\sqrt{2}$ by 1.414. Thus, the exact value $x = \sqrt{2}$ and approximate value is $c = 1.414$. Therefore,

$$\text{Absolute error} = |c - x| = |1.414 - \sqrt{2}| \quad \text{and} \quad \text{Relative error} = \frac{|1.414 - \sqrt{2}|}{|\sqrt{2}|}$$
$$\approx 0.0002135 \quad \approx 0.00015101$$

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10.2 Integral Approximation

In this part, we introduce Midpoint, Trapezoid and Simpson's Rule to approximate the integral $\int_a^b f(x)dx$.

10.2.1 Midpoint Rule ($M(n)$)

Suppose f is defined and integrable on $[a, b]$. The Midpoint Rule approximation to $\int_a^b f(x)dx$ using n equally spaced subintervals on $[a, b]$ is

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \dots + f(m_n)\Delta x = \sum_{i=1}^n f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$, and $m_i = \left(\frac{x_{i-1}+x_i}{2}\right)$ is midpoint of $[x_{i-1}, x_i]$, for $i = 1, \dots, n$.

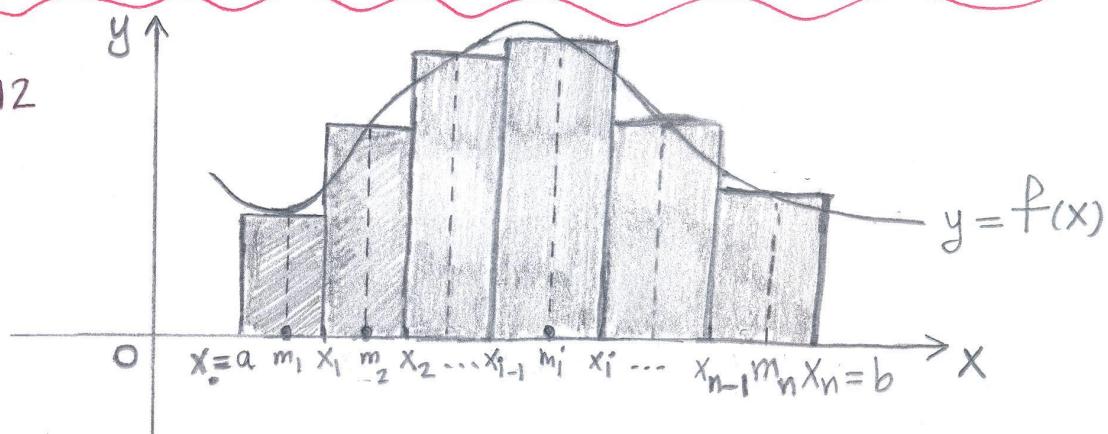
In summary,

$$\begin{aligned} \int_a^b f(x)dx &\approx M(n) = f\left(\frac{x_0+x_1}{2}\right)\Delta x + f\left(\frac{x_1+x_2}{2}\right)\Delta x + \dots + f\left(\frac{x_{n-1}+x_n}{2}\right)\Delta x \\ &= \sum_{i=1}^n f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x, \end{aligned}$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$ $i = 1, \dots, n$.

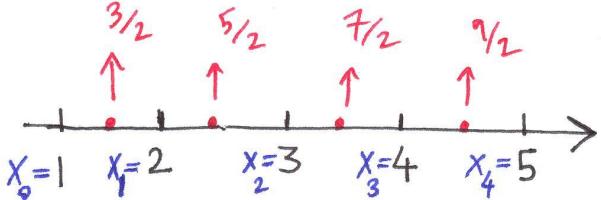
Figure 7.12

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Example 2: Approximate $\int_1^5 x^3 dx$ using the Midpoint Rule with $n=4$.

$$\int_1^5 x^3 dx \approx M_4 = \Delta x \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_4}{2}\right) \right]$$



Therefore,

$$\begin{aligned} M_4 &= 1 \left[f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right) + f\left(\frac{9}{2}\right) \right] \\ &= 1 \left[\left(\frac{3}{2}\right)^3 + \left(\frac{5}{2}\right)^3 + \left(\frac{7}{2}\right)^3 + \left(\frac{9}{2}\right)^3 \right] \\ &= 153 \end{aligned}$$

so,

$$\int_1^5 x^3 dx \approx M_4 = \boxed{153} \quad \text{→ approximate value "c"}$$

$$\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = \frac{4}{4} = 1$$

$$x_i = a + i \Delta x \quad i=0, 1, 2, 3, 4$$

$$x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4,$$

$$x_4 = 5$$

$$m_1 = \frac{x_0+x_1}{2} = \frac{2+1}{2} = \frac{3}{2}$$

$$m_2 = \frac{x_1+x_2}{2} = \frac{3+2}{2} = \frac{5}{2}$$

$$m_3 = \frac{x_2+x_3}{2} = \frac{4+3}{2} = \frac{7}{2}$$

$$m_4 = \frac{x_3+x_4}{2} = \frac{5+4}{2} = \frac{9}{2}$$

$$\begin{aligned} \text{Exact value (solution)} &= \int_1^5 x^3 dx = \frac{x^4}{4} \Big|_1^5 = \frac{5^4}{4} - \frac{1^4}{4} \\ &= \frac{5^4 - 1}{4} = \frac{625 - 1}{4} = \frac{624}{4} = \boxed{156} \rightarrow \text{exact value "x"} \end{aligned}$$

$$\text{Absolute Error} = |c - x| = |153 - 156| = |-3| = 3$$

$$\text{Relative Error} = \frac{|c-x|}{|x|} = \frac{3}{156} = 0.0192307$$

③ 10.2.2 The Trapezoid Rule ($T(n)$)

Suppose f is defined and integrable on $[a, b]$. The Trapezoid Rule approximation to $\int_a^b f(x) dx$ using n equally spaced subintervals on $[a, b]$ is

$$T(n) = \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n) \right],$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, for $i=0, 1, \dots, n$.

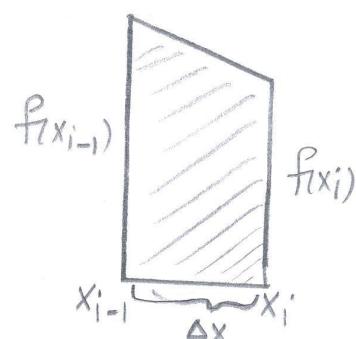
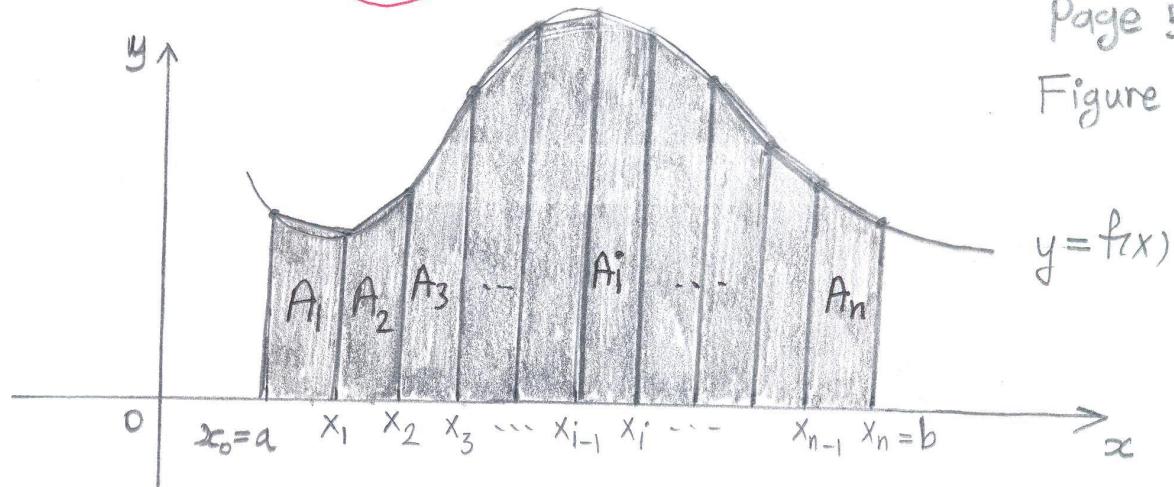
So,

$$\int_a^b f(x) dx \approx T(n) = \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n) \right]$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$, for $i=0, 1, \dots, n$

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Figure 7.14



$$A_i = \frac{f(x_{i-1}) + f(x_i)}{2} \Delta x$$

Note: To remember the formula, note that the coefficients start and end with 1 and others have 2. $\{1, 2, 2, \dots, 2, 2, 1\}$

$$T(n) = \frac{\Delta x}{2} [1f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + 1f(x_n)]$$

Example 3: Approximate $\int_1^5 x^3 dx$ using the Trapezoid Rule with $n=4$. From example 2, we have $\Delta x = 1$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$! So,

$$\begin{aligned}\int_1^5 x^3 dx &\approx T_4 = \frac{1}{2} [f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)] \\ &= \frac{1}{2} [(1)^3 + 2(2)^3 + 2(3)^3 + 2(4)^3 + 5^3] \\ &= \frac{1}{2} [2 + 16 + 54 + 128 + 125] \\ &= 162\end{aligned}$$

Thus,

$$\int_1^5 x^3 dx \approx T_4 = 162$$

$$\text{Exact value (solution)} = 156$$

$$\text{Absolute Error} = |162 - 156| = 6$$

$$\text{Relative Error} = \frac{6}{156} = 0.038461$$

④ 10.2.3 Simpson's Rule ($S(n)$)

An improvement over the Midpoint and Trapezoid Rule results when the graph of f is approximated with curves rather than line segments.

Indeed, in Simpson's Rule, we approximate the integral by parabolas. It is much more complicated, so, we won't graph it here.

- Suppose f is defined and integrable on $[a, b]$ and $n \geq 2$ is an even integer. The Simpson's Rule approximation to $\int_a^b f(x) dx$ using n equally spaced subintervals on $[a, b]$ is

$$\int_a^b f(x) dx \approx S(n) = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 4f(x_{n-1}) + f(x_n)),$$

where n is an even integer, $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\Delta x$, for $i = 0, 1, \dots, n$.

Note 1

The Simpson's Rule results are better approximation than the Midpoint and Trapezoid results. → "most of the time (not always)"

Note 2

That is a complicated formula, but just remember the coefficients start and end with 1 and alternate between 4's and 2's in the middle. $\{1, 4, 2, 4, 2, \dots, 2, 4, 1\}$

$$S(n) = \frac{\Delta x}{3} (1f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + 1f(x_n))$$

Example 4: Approximate $\int_1^5 x^3 dx$ using the Simpson's rule with $n=4$. Again using the information in Example 1 & 2:

$$\begin{aligned}\int_1^5 x^3 dx \approx S_4 &= \frac{1}{3} [f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)] \\ &= \frac{1}{3} [(1)^3 + 4(2)^3 + 2(3)^3 + 4(4)^3 + 5^3] \\ &= 156\end{aligned}$$

Exact value (solution) = 156

$$\text{Absolute Error} = |156 - 156| = 0 \quad \text{Relative Error} = \frac{0}{156} = 0$$

Example 5: Approximate $\int_1^3 \ln x dx$ using the Midpoint, Trapezoid and Simpson's Rules with $n=8$. Which one is the better approximation?

Let's start by finding Δx , $x_i = a + i\Delta x$ ($i=0, 1, \dots, 8$) and

$m_i = \frac{x_{i-1} + x_i}{2}$ ($i=1, 2, 3, \dots, 8$). We know $b=3$ and $a=1$.

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{8} = \frac{2}{8} = \frac{1}{4} \Rightarrow \boxed{\Delta x = \frac{1}{4}}$$

$$x_i = 1 + i(\Delta x) = 1 + i\left(\frac{1}{4}\right) = 1 + \frac{i}{4} \Rightarrow \boxed{x_0 = 1, x_1 = \frac{5}{4}, x_2 = \frac{3}{2},}$$

$$\boxed{x_3 = \frac{7}{4}, x_4 = 2, x_5 = \frac{9}{4}, x_6 = \frac{5}{2}, x_7 = \frac{11}{4}, x_8 = 3}$$

$$m_1 = \frac{x_0 + x_1}{2} = \frac{1 + \frac{5}{4}}{2} = \frac{9}{8}$$

$$m_5 = \frac{x_4 + x_5}{2} = \frac{2 + \frac{9}{4}}{2} = \frac{17}{8}$$

$$m_2 = \frac{x_1 + x_2}{2} = \frac{\frac{5}{4} + \frac{3}{2}}{2} = \frac{11}{8}$$

$$m_6 = \frac{x_5 + x_6}{2} = \frac{\frac{9}{4} + \frac{5}{2}}{2} = \frac{19}{8}$$

$$m_3 = \frac{x_2 + x_3}{2} = \frac{\frac{3}{2} + \frac{7}{4}}{2} = \frac{13}{8}$$

$$m_7 = \frac{x_6 + x_7}{2} = \frac{\frac{5}{2} + \frac{11}{4}}{2} = \frac{21}{8}$$

$$m_4 = \frac{x_3 + x_4}{2} = \frac{\frac{7}{4} + 2}{2} = \frac{15}{8}$$

$$m_8 = \frac{x_7 + x_8}{2} = \frac{\frac{11}{4} + 3}{2} = \frac{23}{8}$$

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• Midpoint:

$$M_8 = \Delta x (f(m_1) + f(m_2) + \dots + f(m_8))$$

$$= \frac{1}{4} \left(\ln\left(\frac{9}{8}\right) + \ln\left(\frac{11}{8}\right) + \ln\left(\frac{13}{8}\right) + \ln\left(\frac{15}{8}\right) + \ln\left(\frac{17}{8}\right) + \ln\left(\frac{19}{8}\right) + \ln\left(\frac{21}{8}\right) + \ln\left(\frac{23}{8}\right) \right)$$

$$= 1.297564$$

• Trapezoid:

$$T_8 = \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_7) + f(x_8))$$

$$\begin{array}{c} \Delta x \\ \uparrow \\ \boxed{\frac{1}{4}} \\ \boxed{\frac{1}{2}} \end{array} = \boxed{\frac{1}{8}} \left(\ln(1) + 2\ln\left(\frac{5}{4}\right) + 2\ln\left(\frac{3}{2}\right) + 2\ln\left(\frac{7}{4}\right) + 2\ln(2) + 2\ln\left(\frac{9}{4}\right) + 2\ln\left(\frac{5}{2}\right) + 2\ln\left(\frac{11}{4}\right) + \ln(3) \right) = 1.292374$$

• Simpson's

$$S_8 = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_6) + 4f(x_7) + f(x_8))$$

$$\begin{array}{c} \Delta x \\ \uparrow \\ \boxed{\frac{1}{3}} \\ \boxed{\frac{11}{4}} \\ \boxed{\frac{1}{3}} \end{array} = \boxed{\frac{1}{12}} \left(\ln(1) + 4\ln\left(\frac{5}{4}\right) + 2\ln\left(\frac{3}{2}\right) + 4\ln\left(\frac{7}{4}\right) + 2\ln(2) + 4\ln\left(\frac{9}{4}\right) + 2\ln\left(\frac{5}{2}\right) + 4\ln\left(\frac{11}{4}\right) + \ln(3) \right) = 1.295798$$

• Exact value

$$\int_1^3 \ln x \, dx = x \ln x - x \Big|_1^3 = (3 \ln 3 - 3) - (1 \ln 1 - 1) = 3 \ln 3 - 2 \approx 1.295836$$

• Absolute Error

$$\text{Error in } M_8 = |\text{approximate value} - \text{exact value}| = |1.297564 - 1.295836| = 0.001728$$

$$\text{Error in } T_8 = |\text{approximate value} - \text{exact value}| = |1.292374 - 1.295836| = 0.003462$$

$$\text{Error in } S_8 = |\text{approximate value} - \text{exact value}| = |1.295798 - 1.295836| = 0.000038$$

"Best approximation" ↪

10.3 Error in Numerical Integration

In general, we want to minimize the error. In the case of our integral approximations we don't always know the actual value. So, we can't compute error explicitly. But, we can find an upper on what the error should be.

Theorem:

- If f'' is continuous on $[a, b]$ and there is a M such that $|f''(x)| \leq M$ for all $x \in [a, b]$, then

$$\textcircled{1} \quad E_M = \left| \int_a^b f(x) dx - M(n) \right| \leq \frac{M(b-a)}{24} (\Delta x)^2 = \frac{M(b-a)^3}{24n^2}$$

$$\textcircled{2} \quad E_T = \left| \int_a^b f(x) dx - T(n) \right| \leq \frac{M(b-a)}{12} (\Delta x)^2 = \frac{M(b-a)^3}{12n^2}$$

- If $f^{(4)}$ is continuous on $[a, b]$ and there is a K such that $|f^{(4)}(x)| \leq K$ for all $x \in [a, b]$, then

$$\textcircled{3} \quad E_S = \left| \int_a^b f(x) dx - S(n) \right| \leq \frac{K(b-a)}{180} (\Delta x)^4 = \frac{K(b-a)^5}{180n^4}.$$

You do not need to memorize these formulas

⑥ Remark:

$$\bullet |E_M| \leq \frac{M(b-a)^3}{24n^2}$$

$$|f''(x)| \leq M$$

$$\bullet |E_T| \leq \frac{M(b-a)^3}{12n^2}$$

$$|f^{(4)}(x)| \leq K$$

$$\bullet |E_S| \leq \frac{K(b-a)^5}{180n^4}$$

Example 6: Let $f(x) = e^{-x^2}$, and consider $\int_0^1 e^{-x^2} dx$.

(a) Use the fact that $|f''(x)| \leq 2$, and $|f^{(4)}(x)| \leq 36$ on the interval $[0,1]$ to estimate the following errors: we have $n=8$, $a=0$, $b=1$.

(Upper bound for $f'' = 2$)

$$\text{Error in } T_8 = |E_T| \leq \frac{M(b-a)^3}{12n^2} = \frac{2(1-0)^3}{12(8)^2} = \frac{1}{6(64)} = 0.0026041$$

$$\text{Error in } M_8 = |E_M| \leq \frac{M(b-a)^3}{24n^2} = \frac{2(1-0)^3}{24(8)^2} = \frac{1}{12(64)} = 0.001302$$

$$\text{Error in } S_8 = |E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{36(1-0)^5}{180(8)^4} = 0.0000488$$

(K = upper bound for $f^{(4)}$, which is given and it's 36)

(b) Using the information in part(a) and the error formulas, how large do we have to choose n so that the approximation T_n , M_n and S_n are accurate to within $10^{-6} = 0.000001$.

For T_n : $n = \boxed{409}$

we know $|E_T| \leq \frac{M(b-a)^3}{12n^2}$. to ensure that $|E_T| \leq 10^{-6}$, we need to have the condition $\frac{M(b-a)^3}{12n^2} \leq 10^{-6}$. we also know that $M=2$, $a=0$, $b=1$. so, we get

$$10^6 \left[\frac{2(1-0)^3}{12n^2} \leq 10^{-6} \right] \Rightarrow \frac{2 \cdot 10^6}{12n^2} \leq 1 \Rightarrow \frac{2 \cdot 10^6}{12} \leq n^2 \Rightarrow \boxed{n \geq 408.248}$$

we choose first integer after this real number

For M_n : $n = \boxed{289}$

Here, the condition is

$$\Rightarrow \frac{2 \cdot 10^6}{24} \leq n^2 \Rightarrow n \geq \frac{10^3}{\sqrt{12}} = 288.67 \rightarrow \begin{array}{l} \text{First integer} \\ \text{greater than} \\ \boxed{n=289} \end{array} \text{ this value}$$

For S_n : $n = \boxed{22}$

Since $K = 36$, for Simpson's rule we have

$$\Rightarrow n^4 \geq \frac{36 \cdot 10^6}{180} \Rightarrow n \geq 21.14 \rightarrow \boxed{n=22}$$

\curvearrowleft First integer greater than this value

Example 7: Let $f(x) = 2 \sin(5x)$, and consider $\int_0^{\pi/2} 2 \sin(5x) dx$.

(a) obtain an upper bound on the absolute error for Simpson's Rule, $|E_S|$, with $n=6$.

(b) Using the information given (obtained) in part(a) and the error formula, how large do we have to choose n to ensure that the approximation S_n is accurate to within 0.0001.

Solution

part(a): $\int_0^{\pi/2} 2 \sin(5x) dx \Rightarrow a=0, b=\frac{\pi}{2}$ and $n=6$. we also have $|E_S| \leq \frac{K(b-a)^5}{180n^4}$. we need to find K , which is an upper bound for the forth derivative of $f(x) = 2 \sin(5x)$. so, we'll find $f^{(4)}(x)$, and find the upper bound on $[0, \frac{\pi}{2}]$.

$$f(x) = 2 \sin(5x) \Rightarrow f'(x) = 2(5)(5 \cos(5x)) \Rightarrow f''(x) = 2(5)^2(-\sin(5x))$$

$$\Rightarrow f'''(x) = 2(5)^3(-\cos(5x)) \Rightarrow \boxed{f^{(4)}(x) = 2(5)^4 \sin(5x)}$$

Now, we found $f^{(4)}(x) = 2(5)^4 \sin(5x)$. We use the fact that $-1 \leq \sin(5x) \leq 1$ on $[0, \frac{\pi}{2}]$. Therefore, $|\sin(5x)| \leq 1$ on $[0, \frac{\pi}{2}]$. So,

$$|f^{(4)}(x)| = |2(5)^4 \sin(4x)| = 2(5)^4 |\sin(4x)| \leq \underbrace{2(5)^4}_K$$

NOW, we have $a=0$, $b=\frac{\pi}{2}$, $K=2(5)^4$ and $n=6$. so,

$$|E_s| \leq \frac{K(b-a)^5}{180 n^4} = \frac{2(5)^4 (\frac{\pi}{2} - 0)^5}{180 \cancel{6^4} \cancel{90}} = \frac{5^4 \cdot (\frac{\pi}{2})^5}{90 \cdot 6^4}$$

Part (b)

We want to have $|E_s| \leq 10^{-4}$. Thus, $\frac{K(b-a)^5}{180 n^4} \leq 10^{-4}$. So,

$$\frac{2(5)^4 (\frac{\pi}{2} - 0)^5}{180 n^4} \leq 10^{-4} \Rightarrow \frac{10^4 (2)(5^4) (\frac{\pi}{2})^5}{180} \leq n^4$$

$$\Rightarrow n \geq 28.54 \rightsquigarrow \boxed{n=29}$$