Integral Calculus

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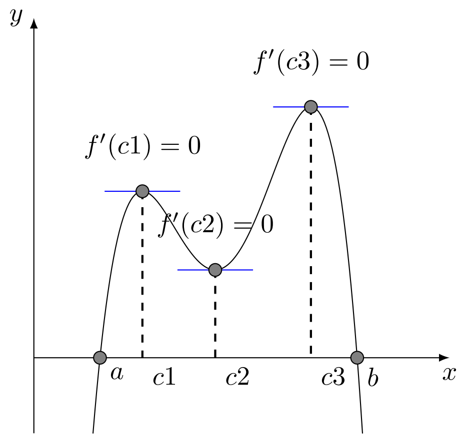
Derivative with two variables (sec. 12.4)

0.1 Recall on derivative with one variable

The **derivative** of a function $y = f(x)$ of a variable x is a measure of the rate at which the value y of the function changes with respect to the change of the variable x. For example, the derivative of the position of a moving object with respect to time is the object's velocity: this measures how quickly the position of the object changes when time is advanced.



The derivative of a function of one variable $y = f(x)$ is defined by:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Notation: $\left.\dfrac{df}{dx}\right|_a = f'(a)$

Example 0.1. $f(x) = x^3 + \sin(x) \Longrightarrow f'(x) = 3x^2 + \cos(x) \Longrightarrow f'(0) = 3(0)^2 + \cos(0) = 1$

0.2 Partial Derivatives

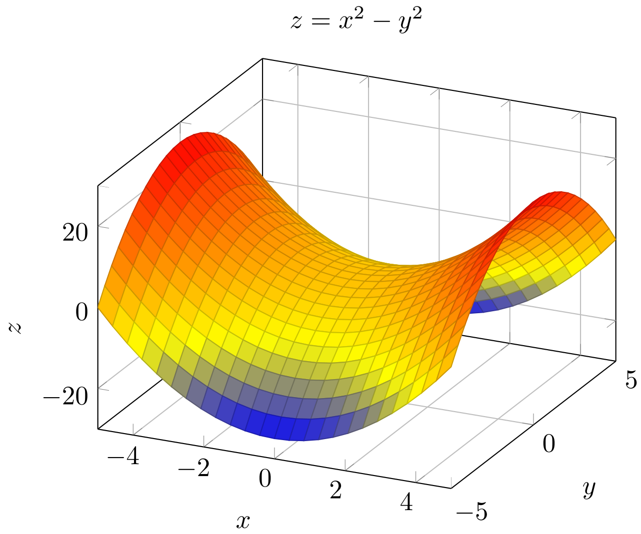
The **partial derivative** of f with **respect to** x at the point (a, b) is:

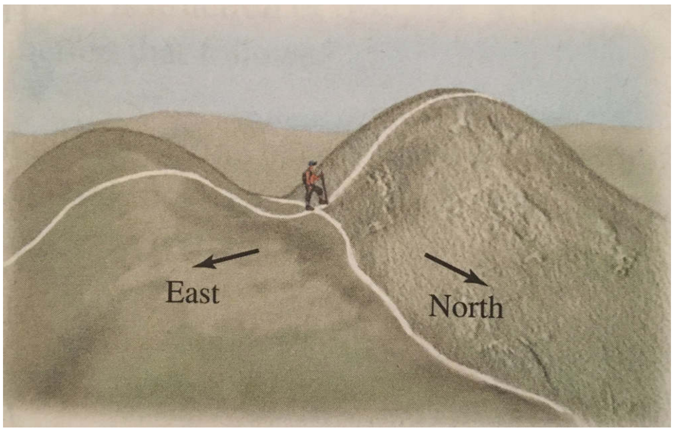
$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

The **partial derivative** of f with **respect to** y at the point (a, b) is:

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Provided these limits exist.





Example 0.2. Example with limit definition. Define $f(x, y)$ by

$$f(x, y) = 2x^2 - 3y^2 - 2$$

If we want to calculate the partial derivative of $f(x, y)$ at any point (x, y), we just have to plug that in the above definition.

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

First, let's compute $f(x + h, y)$:

$$f(x, y) = 2(x + h)^2 - 3y^2 - 2$$

So,

$$\begin{aligned}
f_x(x, y) &= \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{2(x + h)^2 - 3y^2 - 2 - (2x^2 - 3y^2 - 2)}{h} \\
f_x(x, y) &= \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - 3y^2 - 2 - (2x^2 - 3y^2 - 2)}{h} \\
f_x(x, y) &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 3y^2 - 2 - 2x^2 + 3y^2 + 2}{h} \\
f_x(x, y) &= \lim_{h \to 0} \frac{4xh + 2h^2}{h} = \lim_{h \to 0} 4x + 2h \\
f_x(x, y) &= 4x
\end{aligned}$$

Other notation

$$\frac{\partial f}{\partial x}(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b) = \partial_x(a, b)$$

$$\frac{\partial f}{\partial y}(a, b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a, b) = \partial_y(a, b)$$

Meaning

$\dfrac{\partial f}{\partial x}$: derivative of f with respect to x while y is **considered as a constant**.

$\dfrac{\partial f}{\partial y}$: derivative of f with respect to y while x is **considered as a constant**.

Example 0.3. Compute the partial derivative in each case for $f(x, y) = \sqrt{x + y^2}$.

$$f_x(x, y) = \frac{\partial_x(x + y^2)}{2\sqrt{x + y^2}} = \frac{1}{2\sqrt{x + y^2}}$$

$$f_y(x, y) = \frac{\partial_y(x + y^2)}{2\sqrt{x + y^2}} = \frac{2y}{2\sqrt{x + y^2}} = \frac{y}{\sqrt{x + y^2}}$$

Example 0.4. Compute the partial derivative in each case for $f(x, y) = x^2y^3$.

$$f_x(x, y) = 2xy^3; \quad f_y(x, y) = 3x^2y^2$$

Example 0.5. Compute the partial derivative in each case for $f(x, y) = x^2e^{xy}$.
This is a composition of 2 functions of the following form: $f(x, y) = u \cdot v$. Remember $(u \cdot v)' = u' \cdot v + u \cdot v'$.
Here $u = x^2$ and $v = e^{xy}$.

$$f_x(x, y) = \partial_x(x^2) \cdot e^{xy} + x^2 \cdot \partial_x(e^{xy}) = 2x \cdot e^{xy} + x^2 \cdot ye^{xy}$$

$$f_y(x, y) = \partial_y(x^2) \cdot e^{xy} + x^2 \cdot \partial_y(e^{xy}) = 0 \cdot e^{xy} + x^2 \cdot xe^{xy} = x^3e^{xy}$$

Example 0.6. Compute the partial derivative in each case for $f(x, y) = 2x + y^3$.

$$f_x(x, y) = 2; \quad f_y(x, y) = 3y^2$$

Example 0.7. Compute the partial derivative in each case for $f(x, y) = \sqrt{x} + \ln(x^2 + y^2)$.
This is a composition of 2 functions of the following form: $f(x, y) = u \cdot v$ where $(u \cdot v)' = u' \cdot v + u \cdot v'$. Here $u = \sqrt{x}$ and $v = \ln(x^2 + y^2)$. Rm: $(\ln w)' = \dfrac{w'}{w}$

$$f_x(x, y) = \partial_x(\sqrt{x}) \cdot \ln(x^2 + y^2) + \sqrt{x} \cdot \partial_x[\ln(x^2 + y^2)] = \frac{1}{2\sqrt{x}}\ln(x^2 + y^2) + \sqrt{x}\frac{\partial_x(x^2 + y^2)}{x^2 + y^2} = \frac{\ln(x^2 + y^2)}{2\sqrt{x}} + \frac{2x\sqrt{x}}{x^2 + y^2}$$

$$f_y(x, y) = \partial_y(\sqrt{x}) \cdot \ln(x^2 + y^2) + \sqrt{x} \cdot \partial_y[\ln(x^2 + y^2)] = 0 \cdot \ln(x^2 + y^2) + \sqrt{x}\frac{\partial_y(x^2 + y^2)}{x^2 + y^2} = \frac{2y\sqrt{x}}{x^2 + y^2}$$

0.3 High-Order Partial Derivative

A second order partial derivative is just the **partial derivative of the first order partial derivative**.

- The partial derivative of $f_x(x, y)$ with respect to x is: $f_{xx}(x, y) = (f_x)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$
- The partial derivative of $f_x(x, y)$ with respect to y is: $f_{xy}(x, y) = (f_x)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$
- The partial derivative of $f_y(x, y)$ with respect to x is: $f_{yx}(x, y) = (f_y)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$

- The partial derivative of $f_y(x, y)$ with respect to y is: $f_{yy}(x, y) = (f_y)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$

Example 0.8. $f(x, y) = x^4 y - 3xy^2$. Find the first partial derivatives and the second partial derivatives.
First order:

$$f_x = 4x^3 - 3y^2; \qquad f_y = x^4 - 6xy$$

Second order:

$$f_{xx} = 12x^2; \qquad f_{xy} = 4x^3 - 6y$$
$$f_{yx} = 4x^3 - 6y; \qquad f_{yy} = -6x$$

$\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$**. Is this always true?**

Theorem 1. **Equality of Mixed Partial Derivatives**. Assume that f is defined on an open set $\mathcal{D}$ of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout $\mathcal{D}$. Then $f_{xy} = f_{yx}$ at all points of $\mathcal{D}$

Example 0.9. Given the following function:

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } \ (x, y) \neq (0, 0) \\ 0 & \text{if } \ (x, y) = (0, 0) \end{cases}$$

Verify the equality of the mixed partial derivatives at $(0, 0)$.
Solution: for $(x, y) \neq 0$, we have:

$$f(x, y) = \frac{x^3 y - xy^3}{x^2 + y^2}$$

First, let's compute the first derivatives:

$$f_x(x, y) = \frac{(3x^2 y - y^3)(x^2 + y^2) - (x^3 y - xy^3)2x}{(x^2 + y^2)^2}$$

Taking $x = 0$ and for $y \neq 0$, we have:

$$f_x(0, y) = \frac{-y^5}{y^4} = -y$$

From the definition of f we can see that $f(x, 0) = 0$ (whether or not $x = 0$). But in particular $f_x(0, 0) = 0$. Hence,

$$f_{xy}(0, 0) = \lim_{y \to 0} \frac{f_x(0, y) - f_x(0, 0)}{y} = \lim_{y \to 0} \frac{-y - 0}{y} = -1$$

Similarly for $(x, y) \neq (0, 0)$,

$$f_y(x, y) = \frac{(x3 - 3xy^2)(x^2 + y^2) - (x^3 y - xy^3)2y}{(x^2 + y^2)^2}$$

Taking $y = 0$, we have:

$$f_y(x, 0) = \frac{x^5}{x^4} = x$$

Further for $x = 0$, we have $f(x, y) = 0$ (whether or not $x = 0$), so $f_y(0, y) = 0$ for any y. In particular, $f_y(0, 0) = 0$. Hence,

$$f_{yx}(0, 0) = \lim_{x \to 0} \frac{f_y(x, 0) - f_y(0, 0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$

Example 0.10. Is it possible to have a function $f(x, y)$ defined every where with $f_x(x, y) = 2x$ and $f_y(x, y) = 2x$?
$f_{xy} = 0$ and $f_{yx} = 2$. So, it is not possible!!!

Example of application of partial derivatives: The production P of a given factory is described as a function of capital investment K (measured in dollars) and labour L (measured in worker hours). $\dfrac{\partial P}{\partial K}$ is the variation of the production P with respect to the capital investment K while maintaining the worker hours L constant. Whereas, $\dfrac{\partial P}{\partial L}$ is the variation of the production P with respect to the worker hours L while maintaining the capital investment K constant.