

Integral Calculus

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(4)

9. Partial Fractions

Suppose $f(x) = \frac{P(x)}{q(x)}$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q . Sometime $\int f(x)dx = \int \frac{P(x)}{q(x)} dx$ is difficult to integrate, in this case, one of the options, is method of partial fractions.

Let's explain more about idea of method of partial fractions with an example. Let $f(x) = \frac{3x}{x^2 + 2x - 8} = \frac{p(x)}{q(x)}$.

Rational Function

$$\frac{3x}{x^2 + 2x - 8}$$

Partial fraction decomposition

method of
Partial fraction

$$\frac{1}{x-2} + \frac{2}{x+4}$$

Difficult to integrate

$$\int \frac{3x}{x^2 + 2x - 8} dx$$

Easy to integrate

$$\int \frac{1}{x-2} dx + \int \frac{2}{x+4} dx$$

The key idea:

It seems to be hard to deal with the integral of the form $\int \frac{P(x)}{Q(x)} dx$, when $P(x)$ and $Q(x)$ are polynomials. But, it turns out by factorization the denominator (if it's possible), we can write $\frac{P(x)}{Q(x)}$ as a easier form. Then, we can deal with the simplified form. For example if we have $\frac{3x}{x^2+2x-8}$, we have $x^2+2x-8 = (x-2)(x+4)$.

and we can write it as $\frac{3x}{x^2+2x-8} = \frac{A}{x-2} + \frac{B}{x+4}$, by partial fraction decomposition. then, by doing some algebra we find A & B, and using the useful formula $\int \frac{dx}{x+a} = \ln|x+a| + C$, we deal with $\int \frac{3x}{x^2+2x-8} dx$!!! In fact, $\frac{3x}{x^2+2x-8} = \frac{A(x+4) + B(x-2)}{(x-2)(x+4)}$. (common denominator)
thus, $3x = (A+B)x + (4A-2B) \Rightarrow \begin{cases} A+B=3 \\ 4A-2B=0 \end{cases} \Rightarrow A=1 \text{ and } B=2$.

$$\text{Therefore, } \int \frac{3x}{x^2+2x-8} dx = \int \frac{1}{x-2} dx + \int \frac{2}{x+4} dx = \ln|x-2| + 2\ln|x+4| + C$$

9.1 Simple Linear Factors (Procedure)

$f(x) = \frac{P(x)}{Q(x)}$, P and Q are polynomials. (degree of P < degree of Q)

Step 1. Factor the denominator Q: $Q(x) = (x-r_1)(x-r_2) \dots (x-r_n)$.

Step 2. Partial fraction decomposition: $\frac{P(x)}{Q(x)} = \frac{A_1}{(x-r_1)} + \frac{A_2}{(x-r_2)} + \dots + \frac{A_n}{(x-r_n)}$.

Step 3. Clear denominator: Multiply both sides of step 2 by $Q(x) = (x-r_1)(x-r_2) \dots (x-r_n)$, which produces conditions for A_1, \dots, A_m .

Step 4. Solve for coefficients: Equate like powers of x to find A_1, \dots, A_m !

⑤ Example 4: Evaluate $\int \frac{3}{x^2-4} dx$

Step 1 Factor the denominator:

$$x^2 - 4 = (x+2)(x-2)$$

Step 2 Partial Fraction Decomposition:

$$\frac{3}{x^2-4} = \frac{3}{(x+2)(x-2)} = \frac{A}{(x+2)} + \frac{B}{x-2} \quad (I)$$

Step 3 Multiply the equality in Step 2, by the denominator. $\nearrow (I)$

$$\left(\frac{3}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2} \right) \cdot (x+2)(x-2)$$

$$\Rightarrow 3 = A(x-2) + B(x+2) \quad (\star)$$

Step 4 Finding A & B.

Method I (standard):

$$\begin{aligned} \text{From } \star: 3 &= A(x-2) + B(x+2) = Ax - 2A + Bx + 2B \\ &= (A+B)x + 2B - 2A \end{aligned}$$

$$\text{so, } \underset{\approx}{\cancel{3}} = \underset{\approx}{\cancel{(A+B)x}} + \underset{\approx}{\cancel{[2B - 2A]}} \Rightarrow \begin{cases} A+B=0 \\ 2B-2A=3 \end{cases}$$

$$\begin{aligned} \Rightarrow \begin{cases} A=-B \\ 2B-2(-B)=3 \end{cases} &\stackrel{\text{plug } A=-B \text{ to } 2B-2A=3}{\Rightarrow} 2B+2B=3 \Rightarrow B=\frac{3}{4} \\ \text{and } A=-B=\frac{-3}{4} &\Rightarrow A=\frac{-3}{4} \end{aligned}$$

Method II (shortcut)

The roots of $x+2$ and $x-2$ are $x=2$ and $x=-2$!

$$\text{Let } x=2 \text{ in } (\star): 3 = A(2-2) + B(2+2)^4 \Rightarrow 3 = 4B \Rightarrow B = \frac{3}{4}$$

$$\text{Let } x=-2 \text{ in } (\star): 3 = A(-2-2) + B(2-2)^4 \Rightarrow 3 = -4A \Rightarrow A = -\frac{3}{4}$$

(Step 5) Evaluate the integral using the simplified form:

$$\frac{3}{x^2-4} = \frac{-\frac{3}{4}}{x+2} + \frac{\frac{3}{4}}{x-2} \quad (\text{By Steps ① to ④})$$

Therefore,

$$\int \frac{3}{x^2-4} dx = \int \frac{-\frac{3}{4}}{x+2} dx + \int \frac{\frac{3}{4}}{x-2} dx$$

$$= -\frac{3}{4} \int \frac{dx}{x+2} + \frac{3}{4} \int \frac{dx}{x-2}$$

$$= \boxed{-\frac{3}{4} \ln|x+2| + \frac{3}{4} \ln|x-2| + C}$$

Useful formula in Partial Fractions:

$$\int \frac{dx}{x+a} = \ln|x+a| + C \quad (a \rightarrow \text{real number})$$

⑥ Example 5: Evaluate $\int \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} dx$

Step 1 Factor the denominator:

$$x^3 - x^2 - 2x = x(x^2 - x - 2) = x((x-2)(x+1)) = x(x+1)(x-2).$$

Step 2 Partial Fraction Decomposition:

$$\frac{3x^2 + 7x - 2}{x(x+1)(x-2)} = \frac{3x^2 + 7x - 2}{-x^2 - 2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2} \quad (I)$$

Step 3 Multiply the equality in step 2 (I) by the denominator:

$$\left(\frac{3x^2 + 7x - 2}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2} \right) \cdot x(x+1)(x-2)$$

$$\rightarrow 3x^2 + 7x - 2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1) \quad (\star)$$

Step 4: Finding A, B & C:

→ "easy one"

Method (I) (shortcut):

Find the roots of $x, x+1 \& x-2$, which are $x=0, x=-1$ and $x=2$.

- Let $x=0$ in (\star) : $3(0)^2 + 7(0) - 2 = A(1)(-2) + B(0)(-2) + C(0)(1) \Rightarrow -2 = -2A \Rightarrow A = +1$
- Let $x=-1$ in (\star) : $3(-1)^2 + 7(-1) - 2 = A(0)(-3) + B(-1)(-3) + C(-1)(0) \Rightarrow -6 = +3B \Rightarrow B = -2$
- Let $x=2$ in (\star) :

$$3(2)^2 + 7(2) - 2 = A(3)(0) + B(2)(0) + C(2)(3) \Rightarrow 24 = 6C$$

$$\Rightarrow C = 4$$

Method II (standard) → "hard one"

From the right side of \star , we get

$$\begin{aligned}
 A(x+1)(x-2) &= A(x^2 - x - 2) = \boxed{A} x^2 \boxed{-A} x \boxed{-2A} \\
 + Bx(x-2) &= B(x^2 - 2x) = \boxed{B} x^2 \boxed{-2B} x \boxed{+0} \\
 + Cx(x+1) &= C(x^2 + x) = \boxed{C} x^2 \boxed{+C} x \boxed{+0}
 \end{aligned}$$

Right Hand Side in \star = $A(x+1)(x-2) + Bx(x-2) + Cx(x+1) = (A+B+C)x^2 + (-A-2B+C)x - 2A$

Left Hand Side in \star = $3x^2 + 7x - 2$

Set them equal to get:

$$3x^2 + 7x - 2 = (A+B+C)x^2 + (-A-2B+C)x - 2A$$

Coefficients of x^2 have to be equal: $A+B+C = 3$

Coefficients of x have to be equal: $-A-2B+C = 7$

The constant terms have to be equal: $-2A = -2$

$$\begin{cases} A+B+C=3 & \textcircled{1} \\ -A-2B+C=7 & \textcircled{2} \end{cases}$$

$$-2A = -2 \rightsquigarrow \boxed{A=1}$$

Now plugging " $A=1$ " into $\textcircled{1}$ and $\textcircled{2}$, we get

$$\begin{aligned}
 \text{from } \textcircled{1} \rightsquigarrow 1+B+C=3 &\rightarrow \begin{cases} B+C=2 & \textcircled{1}' \\ 2B-C=-8 & \textcircled{2}' \end{cases} \\
 \text{from } \textcircled{2} \rightsquigarrow -1-2B+C=7 &\rightarrow
 \end{aligned}$$

Now, by solving this system we have $3B = -6$ (add $\textcircled{1}'$ and $\textcircled{2}'$) $\Rightarrow \boxed{B=-2}$

Finally, plug " $B=-2$ " into $\textcircled{1}'$ to get

$$-2+C=2 \Rightarrow \boxed{C=4}$$

(7)

Step 5

Evaluate the integral using the simplified form:

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{1}{x} + \frac{-2}{x+1} + \frac{4}{x-2} \quad (\text{By Step 1-4})$$

Therefore,

$$\begin{aligned} \int \frac{3x^2+7x-2}{x^3-x^2-2x} dx &= \int \frac{1}{x} dx - 2 \int \frac{1}{x+1} dx + 4 \int \frac{1}{x-2} dx \\ &= \boxed{\ln|x| - 2 \ln|x+1| + 4 \ln|x-2| + C} \end{aligned}$$

9.2 Repeated linear Factors

Let $f(x) = \frac{p(x)}{q(x)}$, p and q are polynomials. Suppose the repeated linear factor $(x-r)^m$ appears in the denominator, that is,

$$q(x) = (x-r)^m (x-r_1)(x-r_2) \dots$$

The partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \dots + \frac{A_m}{(x-r)^m},$$

where A_1, A_2, \dots, A_m are constants to be determined.

In fact, we have

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x-r)^m (x-r_1)(x-r_2)\dots} = \frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \dots + \frac{A_m}{(x-r)^m} + \frac{B}{x-r_1} + \frac{C}{x-r_2} + \dots$$

we start with $(x-r)$ with power "one" and finish with power " m !!!"

Example 6 Evaluate $\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2} dx$

Step 1 Factor the denominator:

$$x^3 - 2x^2 = x^2(x-2)$$

Step 2 Partial Fraction Decomposition:

$$\frac{5x^2 - 3x + 2}{x^3 - 2x^2} = \frac{5x^2 - 3x + 2}{\cancel{x^2}(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \quad (\text{I})$$

↳ "we start with x and finish with x^2 "

Step 3 Multiply (I) by the denominator:

$$\left(\frac{5x^2 - 3x + 2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \right) \cdot x^2(x-2)$$

$$\Rightarrow 5x^2 - 3x + 2 = A(x-2) + B(x-2) + Cx^2 \quad (\star)$$

Step 4 Finding A, B & C:

Method I (shortcut) :

The roots of x^2 and $x-2$ are $x=0$ (repeated) and $x=2$.

• Let $x=0$ in (\star) : $5(0)^2 - 3(0) + 2 = A(0)(-2) + B(-2) + C(0)^2$
 $\Rightarrow 2 = -2B \Rightarrow B = -1$

• Let $x=2$ in (\star) : $5(2)^2 - 3(2) + 2 = A(2)(0) + B(0)^2 + C(2)^2$
 $\Rightarrow 16 = 4C \Rightarrow C = 4$

⑧ since $x=0$ is repeated root, there is no more root to plug into (\star) to find A !!! So, to deal with this problem we substitute $B = -1$ and $C = 4$ that we found in (\star)

$B = -1$ & $C = 4$ into (\star) : \rightarrow

$$5x^2 - 3x + 2 = Ax(x-2) - (x-2) + 4x^2 \quad (\star\star)$$

Now, we use the fact that $(\star\star)$ has to be true for all values of x . So, we can choose x any numbers and plug into $(\star\star)$ to find A . I choose $x = 1$: \rightarrow

Let $x = 1$ in $(\star\star)$:

$$5(1)^2 - 3(1) + 2 = A(1)(-1) - (1-2) + 4(1) \Rightarrow$$

$$4 = -A + 1 + 4 = -A + 5 \Rightarrow$$

$$-1 = -A \Rightarrow \boxed{A = 1}$$

Note: ↴

Now, let say you choose $x = -2$ (different from my choice)
we get (which was $x = 1$)

$x = -2$ in $(\star\star)$:

$$5(-2)^2 - 3(-2) + 2 = A(-2)(-4) - (-2-2) + 4(-2)^2 \Rightarrow$$

$$28 = 8A + 4 + 16 \Rightarrow$$

$$28 = 8A + 20 \Rightarrow$$

$$\therefore 8 = 8A \Rightarrow \boxed{A = 1}$$

"you can see we get the same A by any choice of x "

not "0" and "2"
we already considered
these values

Method II (Standard)

From (*) we have

$$5x^2 - 3x + 2 = A(x^2 - 2x) + BX - 2B + CX^2 \Rightarrow$$

$$5x^2 - 3x + 2 = Ax^2 - 2Ax + BX - 2B + CX^2 \Rightarrow$$

$$5x^2 - 3x + 2 = (A+C)x^2 + (-2A+B)x - 2B \Rightarrow \begin{cases} A+C=5 \\ -2A+B=-3 \\ -2B=2 \end{cases}$$

$$\Rightarrow \begin{cases} A=5-C \\ 2A=B+3 \\ B=-1 \end{cases} \Rightarrow \begin{cases} A=5-C \\ A=\frac{B+3}{2} \\ B=-1 \end{cases} \Rightarrow \begin{cases} A=5-C \\ A=\frac{-1+3}{2}=1 \\ B=-1 \end{cases}$$

$$\Rightarrow \begin{cases} C=5-1 \\ A=1 \\ B=-1 \end{cases} \Rightarrow \begin{cases} C=4 \\ A=1 \\ B=-1 \end{cases}$$

Step 5

Evaluate the integral using the simplified form:

$$\int -x^{-2} dx = -(-x^{-1}) = \frac{1}{x}$$

$$\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2} dx = \int \frac{1}{x} dx + \int \frac{-1}{x^2} dx + \int \frac{4}{x-2} dx$$

$$= \boxed{\ln|x| + \frac{1}{x} + 4 \ln|x-2| + C}$$

⑨ Example 7: Evaluate $\int \frac{16x^2}{(x-6)(x+2)^2} dx$

Step 1 Factor the denominator:

This one is already done!!! So, $(x-6)(x+2)^2$.

Step 2 Partial Fraction Decomposition:

$$\frac{16x^2}{(x-6)(x+2)^2} = \frac{A}{x-6} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \quad (\text{I})$$

we start by $(x+2)$ and finish with $(x+2)^2$

Step 3 Multiply (I) by the denominator:

$$\left(\frac{16x^2}{(x-6)(x+2)^2} = \frac{A}{x-6} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \right) \cdot (x-6)(x+2)^2 \Rightarrow$$

$$16x^2 = A(x+2)^2 + B(x-6)(x+2) + C(x-6) \quad (\star)$$

Step 4 Finding A, B & C:

we use "the shortcut method" which is much easier here!

Roots of $x-6$ and $(x+2)^2$ are $x=6$ (simple) and $x=-2$ (repeated)

$$\text{Let } x=6 \text{ in } (\star): 16(6)^2 = A(8)^2 + B(0)(8) + C(0) \Rightarrow 16(36) = 64A$$

$$\Rightarrow A = \frac{16 \cdot 36}{64} = \frac{36}{4} = 9 \Rightarrow \boxed{A=9}$$

$$\text{Let } x = -2 \text{ in } (\star): 16(-2)^2 = A(0)^2 + B(-8)(0) + C(-8) \Rightarrow$$

$$16 \cdot 4 = -8C \Rightarrow C = -\frac{16(4)}{8} \Rightarrow \boxed{C=-8}$$

To find B, we plug A=9 and C=-8 into (\star) to get

$$16x^2 = 9(x+2)^2 + B(x-6)(x+2) - 8(x-6) \quad (\star\star)$$

"Note that $(\star\star)$ has to be true for any value of x."

Since $(\star\star)$ is true for any x , we can choose " $x=0$ ". → "don't choose 6 or -2 we already used them"
 let $x=0$ in $(\star\star)$: $16(0)^2 = 9(2)^2 + B(-6)(2) - 8(-6)$

$$\Rightarrow 0 = 36 - 12B + 48 \Rightarrow 12B = 84 \Rightarrow B = \frac{84}{12} \Rightarrow \boxed{B=7}$$

"You can check that by choosing other value x , like 2, 1, ..., you get $B=7$ "

Step 5 Evaluate the integral using the simplified form: "use substitution"

$$\begin{aligned} \int \frac{16x^2}{(x-6)(x+2)^2} dx &= \int \frac{9}{x-6} dx + \int \frac{7}{x+2} dx + \int \frac{-8}{(x+2)^2} dx \\ &= 9 \ln|x-6| + 7 \ln|x+2| + 8 \int \frac{dx}{(x+2)^2} \\ &= \boxed{9 \ln|x-6| + 7 \ln|x+2| - \frac{8}{x+2} + C} \end{aligned}$$

$$\begin{aligned} u &= x+2 \\ du &= dx \\ \int \frac{du}{u^2} &= \int u^{-2} du \\ &= -u^{-1} = -\frac{1}{u} \\ &= \frac{1}{x+2} + C \end{aligned}$$

Note (Regarding Step 4 in Example 7): if we use the standard method, we get (it is really harder than short cut in this case): from (\star) , we have

$$16x^2 = A(x^2 + 4x + 4) + B(x^2 - 4x - 12) + C(x-6) = Ax^2 + 4Ax + 4A + Bx^2 - 4Bx - 12B +$$

$$Cx - 6C = (A+B)x^2 + (4A - 4B + C)x + 4A - 12B - 6C. \text{ Therefore,}$$

$$\boxed{16x^2 + 0x + 0 = (A+B)x^2 + (4A - 4B + C)x + [4A - 12B - 6C]}. \text{ So,}$$

$$\begin{cases} A+B=16 & \textcircled{1} \\ 4A-4B+C=0 & \textcircled{2} \\ 4A-12B-6C=0 & \textcircled{3} \end{cases} \quad \begin{aligned} \text{From } \textcircled{2} \rightarrow C = 4B - 4A. \text{ plug into } \textcircled{3} \rightarrow 4A - 12B - 6(4B - 4A) \\ \rightarrow 4A - 12B - 24B + 24A = 0 \rightarrow 28A - 36B = 0 \quad \textcircled{4} \end{aligned}$$

So, from $\textcircled{1}$ and $\textcircled{4}$, we get $\begin{cases} A+B=16 & \textcircled{1} \\ 28A-36B=0 & \textcircled{4} \end{cases}$. From $\textcircled{4}$, $A = \frac{36}{28} = \frac{9}{7} B$.

plug " $A = \frac{9}{7} B$ " into $\textcircled{1}$, we get $\frac{9}{7} B + B = 16 \Rightarrow \frac{16B}{7} = 16 \Rightarrow \boxed{B=7}$. So,
 $A = \frac{9}{7} B = \frac{9}{7}(7) = 9 \Rightarrow \boxed{A=9}$. plugging " $B=7$ " and " $A=9$ " into

$$\textcircled{2}, \text{ we have } 4(9) - 4(7) + C = 0 \Rightarrow 36 - 28 + C = 0 \Rightarrow 8 + C = 0$$

$\Rightarrow \boxed{C=-8}$ * "You can see this method is much harder and too long" *

①

Lecture note 21

Feb 29, 2016

9.3 Partial Fractions (Long Division)

Here, we deal with the integral of form $\int f(x)dx$, where $f(x) = \frac{P(x)}{q(x)}$,

and P and q are polynomials and also $\boxed{\text{degree } P(x) > \text{degree } q(x)}$.

In this case, since degree of $P(x)$ is greater than degree of $q(x)$, we apply "long division" to write $\frac{P(x)}{q(x)} = h(x) + \frac{r(x)}{q(x)}$. Now, in

$\frac{r(x)}{q(x)}$, degree of $r(x) <$ degree of $q(x)$. So, we can deal with $\frac{r(x)}{q(x)}$, using the methods that we learned in sections 9.1 & 9.2!!

In summary:

$$\int f(x)dx = \int \frac{P(x)}{q(x)} dx \quad (\text{degree } P(x) > \text{degree } q(x))$$

Perform "Long Division:

$$\begin{array}{r} h(x) \\ \hline q(x) \overline{)P(x)} \\ \vdots \\ \hline \vdots \\ \hline r(x) \end{array}$$

$$\frac{P(x)}{q(x)} = h(x) + \frac{r(x)}{q(x)}$$

Therefore,

$$\begin{aligned} \int f(x)dx &= \int \frac{P(x)}{q(x)} dx = \int \left[h(x) + \frac{r(x)}{q(x)} \right] dx \\ &= \int h(x) dx + \int \frac{r(x)}{q(x)} dx \end{aligned}$$

polynomial;
easy to
integrate

apply partial fractions
in 9.1 & 9.2

Example 1: $\int \frac{3x^3 + x^2 - 2x + 10}{x^2 + x - 2} dx$

degree is 3 ✓
degree is 2

So, by using the "Long Division", we get

$$\begin{array}{r} 3x - 2 \\ \hline x^2 + x - 2 \) 3x^3 + x^2 - 2x + 10 \\ - (3x^3 + 3x^2 - 6x) \\ \hline -2x^2 + 4x + 10 \\ - (-2x^2 - 2x + 4) \\ \hline 6x + 6 \end{array}$$

Now, we can rewrite the integrand as

$$\frac{3x^3 + x^2 - 2x + 10}{x^2 + x - 2} = \underbrace{3x - 2}_{\substack{\text{polynomial;} \\ \text{easy to} \\ \text{integrate}}} + \frac{6x + 6}{x^2 + x - 2}$$

degree 1
degree 2

Let's use method of partial fraction to write $\frac{6x+6}{x^2+x-2}$ as

$$\frac{6x+6}{x^2+x-2} = \frac{A}{(x+2)} + \frac{B}{x-1} .$$

Note that $x^2 + x - 2 = (x+2)(x-1)$. So,

$$\frac{6x+6}{x^2+x-2} = \frac{6x+6}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} .$$

(2) By multiplying $(x+2)(x-1)$ into both sides of the last equality, we get

$$(*) \quad 6x+6 = A(x-1) + B(x+2)$$

Roots of $(x+2)$ and $(x-1)$ are $x=-2$ and $x=1$. So,

$$x=1 \text{ in } (*): \quad 6(1)+6 = A(0) + B(3) \Rightarrow 12 = 3B \Rightarrow B=4$$

$$x=-2 \text{ in } (*): \quad 6(-2)+6 = A(-3) + B(0) \Rightarrow -6 = -3A \Rightarrow A=2$$

Therefore,

$$\frac{6x+6}{(x+2)(x-1)} = \frac{2}{x+2} + \frac{4}{x-1}$$

and

$$\frac{3x^3+x^2-2x+10}{x^2+x-2} = 3x-2 + \frac{2}{x+2} + \frac{4}{x-1}.$$

Finally,

$$\begin{aligned} \int \frac{3x^3+x^2-2x+10}{x^2+x-2} dx &= \int (3x-2) dx + \int \frac{2}{x+2} dx + \int \frac{4}{x-1} dx \\ &= \frac{3x^2}{2} - 2x + 2 \ln|x+2| + 4 \ln|x-1| + C \end{aligned}$$

$$= \boxed{\frac{3}{2}x^2 - 2x + 2 \ln|x+2| + 4 \ln|x-1| + C}$$

Example 2: Evaluate $\int \frac{x^4+2x+7}{x^2+3x+2} dx$

So, we use "Long Division" to get

$$\begin{array}{r} x^2 - 3x + 7 \\ x^2 + 3x + 2 \overline{)x^4 + 2x + 7} \\ - (x^4 + 3x^3 + 2x^2) \\ \hline -3x^3 - 2x^2 + 2x + 7 \\ - (-3x^3 - 9x^2 - 6x) \\ \hline 7x^2 + 8x + 7 \\ - (7x^2 + 21x + 14) \\ \hline -13x - 7 \end{array}$$

Step 1

degree is 4

4 > 2.

degree is 2

Step 2: Partial Fractions

Decomposition to $\frac{13x+7}{x^2+3x+2}$.

$$x^2 + 3x + 2 = (x+1)(x+2)$$

$$\frac{13x+7}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} \quad (\text{I})$$

Multiplying (I) by the denominator $(x+2)(x+1)$, we get

$$13x+7 = A(x+2) + B(x+1) \quad (\star)$$

The roots of $(x+1) \& (x+2)$ are $x=-1$ and $x=-2$.

Let $x=-1$ in (\star) :

$$-13+7 = A(1) + B(0)$$

$$\Rightarrow A = -6$$

Let $x=-2$ in (\star) :

$$-26+7 = A(0) + B(-1)$$

$$\Rightarrow -19 = -B \Rightarrow B = 19$$

Step 3 Evaluate the integral using

Step 1 & 2. By Step 1 and 2, we can

rewrite $\frac{x^4+2x+7}{x^2+3x+2}$ as:

$$\frac{x^4+2x+7}{x^2+3x+2} = x^2 - 3x + 7 - \left(\frac{-6}{x+1} + \frac{19}{x+2} \right) = x^2 - 3x + 7 + \frac{6}{x+1} - \frac{19}{x+2}.$$

Thus,

$$\int \frac{x^4+2x+7}{x^2+3x+2} dx = \int (x^2 - 3x + 7) dx + 6 \int \frac{dx}{x+1} - 19 \int \frac{dx}{x+2} = \boxed{\frac{x^3}{3} - \frac{3}{2}x^2 + 7x + 6 \ln|x+1| - 19 \ln|x+2| + C}$$