Integral Calculus

Daniel Rakotonirina

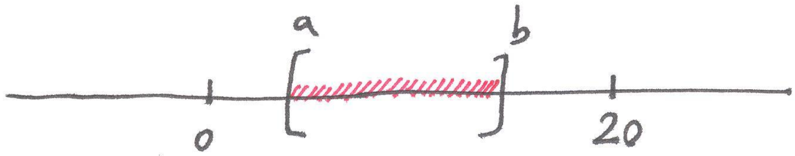
March 16, 2017

Probability

☆ Probability ☆

1. Continuous random variable

Let's start with an example. Suppose that you take a bus to work, and that every 20 minutes a bus arrives at your stop. Because of variation in the time that you leave your home, you don't always arrive at the bus stop at the same time, so your waiting time X for the next bus takes on any valve in the interval [0, 20]. What is the probability (the chance) of your waiting time on any interval [a, b]?



$$= \frac{\text{Length of } [a,b]}{\text{Length of } [0,20]}$$

our guess (after a bit of thought) would be "$\frac{b-a}{20}$".

So,

- The probability of your waiting time on any interval $[a,b]$ $= \frac{b-a}{20}$

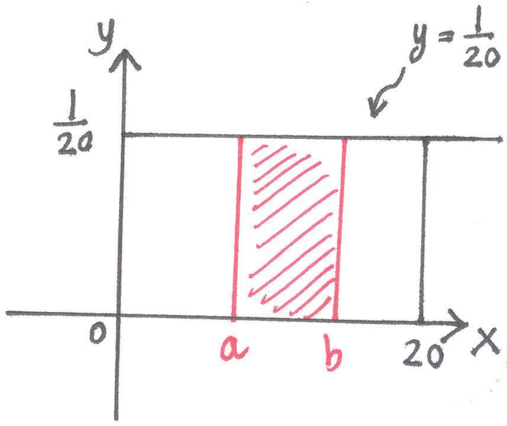
<u>Notation</u>:

$$P(a \le X \le b) = \frac{b-a}{20}.$$

- Since your waiting time X is a random event, the variable X is called a <u>random variable</u>.

② we now consider the function $f(x) = \frac{1}{20}$ with the domain $[0, 20]$. The probability $P(a \le X \le b)$, which is the waiting time on the interval $[a, b]$ is the area under the graph of $y = f(x) = \frac{1}{20}$ between a and b. Therefore,

$$P(a \le X \le b) = \text{the red area}$$
$$= \int_a^b f(x)\,dx$$



So, we get

$$P(a \le X \le b) = \int_a^b f(x)\,dx = \int_a^b \frac{1}{20}\,dx = \frac{x}{20}\Big|_a^b = \frac{b-a}{20}.$$

The function $f(x) = \frac{1}{20}$, $a \le X \le b$, (in this example) is called the probability density function (PDF) of the random variable X.

In General, we have the following definitions:

Definition (CDF):

The cumulative distribution function of any random variable X is the function $F(x) = P(X \le x)$.

Definition (Continuous Random Variable)

A random variable X is a continuous random variable if its cumulative distribution function $P(X \leq x)$ is a continuous function of x.

Definition (PDF):

Let X be a continuous random variable with CDF $F(x)$. If $F(x)$ is differentiable then $f(x) = \frac{dF(x)}{dx}$ is called with probability density function (often shortened to PDF) of X.

Relation Between CDF and PDF:

CDF

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

PDF

$$f(x) = \frac{dF(x)}{dx}$$

So,

① If we have PDF, $f(x)$, to find CDF, we take integral from $-\infty$ to x of $f(x)$.

$$f(x) \text{ (PDF) given} \rightsquigarrow F(x) \text{ (CDF)} = \int_{-\infty}^{x} f(t)\,dt$$

3

② If we have CDF, $F(x)$, to find PDF, we take derivative of $F(x)$ with respect to x.

$F(x)$ (CDF) given $\rightsquigarrow$ $f(x)$ (PDF) $= \dfrac{dF(x)}{dx}$.

Properties of the PDF

Let $f(x)$ be a probability density function (PDF) of a random Variable X. Then,

1. $f(x) \geq 0$ for all x,
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$,
3. (CDF) $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$ (CDF)
4. $P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx$.

★Note★

To verify that a function $f(x)$ is PDF, we have to check Propeties 1 & 2 (above).

$f(x) \geq 0 \ \forall x$ $\quad$ $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Example 1: Let $f(x) = \begin{cases} K(3x^2+1) & \text{if } 0 \le X \le 2 \\ 0 & \text{otherwise} \end{cases}$.

1. Find the value of K that makes the given function a PDF. we need to check ① $f(x) \ge 0$ for all x.
② $\int_{-\infty}^{+\infty} f(x)\,dx = 1$. For ①, $K(\underbrace{3x^2+1}_{\text{always} > 0}) \ge 0 \Rightarrow \boxed{K \ge 0}$

For ②:

$f(x) = 0 \quad x > 2$ or $x < 0$

$$1 = \int_{-\infty}^{+\infty} f(x)\,dx = \int_0^2 K(3x^2+1)\,dx = K\int_0^2 (3x^2+1)\,dx = K \left. x^3+x \right|_0^2$$

$$= K(8+2-0) = 10K \Rightarrow 10K = 1 \Rightarrow \boxed{K = \frac{1}{10}}$$

it is also satisfies ①

2. let X be a continuous random variable whose PDF is f(x). Compute the probability that X is between 1 and 2.

$P(1 \le X \le 2) = ?$

$P(1 \le X \le 2)$.

$$P(1 \le X \le 2) = \int_1^2 \frac{1}{10}(3x^2+1)\,dx = \frac{1}{10}\int_1^2 (3x^2+1)\,dx$$

$$= \frac{1}{10} \left. x^3+x \right|_1^2 = \frac{1}{10}\left((8+2)-(1+1)\right) = \frac{1}{10}(10-2) = \frac{8}{10} = \boxed{\frac{4}{5}}$$

Note that $f(x) = \begin{cases} \frac{1}{10}(3x^2+1) & \text{if } 0 \le X \le 2 \\ 0 & \text{otherwise} \end{cases}$

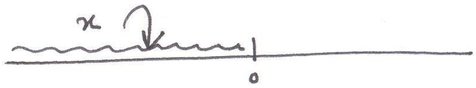
we find K in part (1)!

4

3. Find the cumulative distribution function of X.

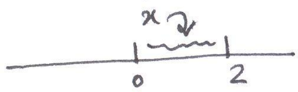
(I) When x is on $(-\infty, 0)$, we have $f(x)=0$ for $x \in (-\infty, 0)$. So,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = \boxed{0}$$



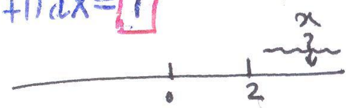
(II) when x is on $[0,2]$: we have $f(x)=0$ for $x\in(-\infty,0)$ and $f(x) = \frac{1}{10}(3x^2+1)$ for $x \in [0,2]$. So, $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int^{x} f(t)\,dt$

$$= \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} \frac{1}{10}(3t^2+1)\,dt = \frac{1}{10}\left. t^3 + t \right|_0^x = \boxed{\frac{x^3+x}{10}}.$$



(III) when x is on $(2,\infty)$: we have $f(x)=0$ for $x\in(-\infty,0)$. $f(x) = \frac{1}{10}(3x^2+1)$ for $x\in[0,2]$, and $f(x)=0$ for $x \in (2,\infty)$. Thus,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} \overset{\to 0}{f(t)}\,dt + \int_{0}^{1} \overset{\to \frac{1}{10}(3x^2+1)}{f(t)}\,dt + \int_{1}^{x} \overset{\to 0}{f(t)}\,dt = \int_{0}^{1} \frac{1}{10}(3x^2+1)\,dx = \boxed{1}$$



$$F(x) = \begin{cases} 0, & x<0 \\ \frac{x^3+x}{10}, & 0 \le x \le 2 \\ 1, & x>2. \end{cases}$$

4. Find the probability that X is exactly equal to 1.

$$P(X=1) = \int_{1}^{1} f(t)\,dt = 0.$$

$$\boxed{P(a \le X \le b) = \int_a^b f(t)\,dt.\ \text{Recall that } a=b \text{ here}}$$

Remark 1: For a continuous random variable X, then the CDF $F(x)$ is continuous, but the PDF $f(x)$ maybe not be continuous.

Remark 2: Not all continuous random variables have PDFs.

Example 2. let $f(x) = \begin{cases} K(x-x^2) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

① Determine K so that f is a probability density function.

We should check ① $f(x) \ge 0$ for all x ② $\int_{-\infty}^{+\infty} f(x)\,dx = 1$.

① since $x - x^2$ on $[0,1]$ is always positive, we need $K \ge 0$.

$K(x-x^2) \ge 0 \implies K \ge 0$

② $1 = \int_{-\infty}^{+\infty} f(x)dx = \int_0^1 K(x-x^2)\,dx = K\int_0^1 x - x^2\,dx = K \left. \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1$

$= K\left(\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{0}{2} - \frac{0}{3}\right)\right) = K\left(\frac{1}{6}\right)$, so, $\int_{-\infty}^{+\infty} f(x)dx = K\left(\frac{1}{6}\right) = 1 \Rightarrow \boxed{K=6}$

② $P(0 \le x \le \frac{1}{2}) = ?$

Note:

$f(x) = \begin{cases} 6(x-x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$P(0 \le X \le \frac{1}{2}) = \int_0^{1/2} 6(x-x^2)\,dx$

$= 6\int_0^{1/2} (x-x^2)\,dx = 6 \left. \frac{x^2}{2} - \frac{x^3}{3} \right|_0^{1/2}$

$= 6\left(\left(\frac{(\frac{1}{2})^2}{2} - \frac{(\frac{1}{2})^3}{3} \right) - 0 \right) = \frac{1}{2}$

③ Find th CDF of X.

(I) if $x \in (-\infty, 0)$: in this case $f(x) = 0$. So, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0\,dt = \boxed{0}$

(II) if $x \in [0,1]$: in this case $f(x) = 0$ for $x \in (-\infty, 0)$ and $f(x) = 6(x-x^2)$ for $x \in [0,1]$.

So, $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} f(t)dt + \int_0^x f(t)dt = 0 + \int_0^x 6(t-t^2)dt$

$= 0 + 3t^2 - 2t^3 \Big|_0^x = \boxed{3x^2 - 2x^3}$

(III) if $x \in (1, \infty)$: $\rightsquigarrow F(x) = \int_{-\infty}^{x} f(t)dt \rightsquigarrow$ split off into 3 parts:

$F(x) = \underbrace{\int_{-\infty}^{0} f(t)dt}_{0} + \underbrace{\int_0^1 f(t)dt}_{6(t-t^2)} + \underbrace{\int_1^{\infty} f(t)dt}_{0} = \int_0^1 6(t-t^2)dt = \boxed{1}$

$F(x) = \begin{cases} 0, & x < 0 \\ 3x^2 - 2x^3, & 0 \le x \le 1 \\ 1, & x \end{cases}$

5

Example 3. Let $f(x) = \begin{cases} K e^{-0.15(x-0.5)} & x \geq 0.5 \\ 0 & \text{otherwise} \end{cases}$

Determine K so that f is a PDF. checking

① $f(x) \geq 0$ for all x. $k e^{-0.15(x-0.5)} \geq 0 \Rightarrow k \geq 0$

since exponential function is always positive

② $\int_{-\infty}^{+\infty} f(x)dx = 1 \Rightarrow 1 = \int_{0.5}^{\infty} k e^{-0.15(x-0.5)} dx$

$= k \int_{\frac{1}{2}}^{\infty} e^{-0.15x + 0.075} dx = \lim_{b\to\infty} k e^{0.075} \int_{\frac{1}{2}}^{b} e^{-0.15x} dx = k e^{0.075} \left. \frac{e^{-0.15x}}{-0.15} \right|_{\frac{1}{2}}^{b}$

$= \frac{k e^{0.075}}{-0.15} \lim_{b\to\infty} \left[\frac{1}{e^{0.15b}} - \frac{1}{e^{0.15(\frac{1}{2})}} \right] = \frac{k \cancel{e^{0.075}}}{0.15 \cancel{e^{0.075}}} \Rightarrow \boxed{k = 0.15}$

Example 4: Let $f(x) = \begin{cases} 0 & x < -1 \\ 1+x & -1 \leq x \leq 0 \\ K(1-x) & 0 < x \leq 1 \\ 0 & x > 1 \end{cases}$.

① Determine K so that f is a PDF.

$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{0} (1+x)dx + \int_{0}^{1} k(1-x)dx$ [Note that $f(x)=0$ on $x \in (-\infty, -1)$ and $f(x)=0$ on $x \in (1, \infty)$]

$= \left. x + \frac{x^2}{2} \right|_{-1}^{0} + \left. k\left(x - \frac{x^2}{2}\right) \right|_{0}^{1} = \left(0 - \left(-1 + \frac{1}{2}\right)\right) + k\left(\left(1 - \frac{1}{2}\right) - 0\right)$

$= \frac{1}{2} + k\left(\frac{1}{2}\right) \Rightarrow \frac{k}{2} + \frac{1}{2} = 1 \Rightarrow \frac{k}{2} = \frac{1}{2} \rightarrow \boxed{k=1}$.

so, choosing $k=1$, we have $f(x) \geq 0$ for all x and also $\int_{-\infty}^{\infty} f(x)dx = 1$.

② Find th CDF of X.

Ⅰ $x \in (-\infty,-1)$: $f(x)=0 \rightsquigarrow F(x)=\int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0\, dt = \boxed{0}$

Ⅱ $x \in [-1,0]$: $f(x)=0$ for $x\in(-\infty,-1)$ and $f(x)=1+x$ $x\in[-1,0)$.

$\rightsquigarrow F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{-1} f(t)dt + \int_{-1}^{x} f(t)dt = 0 + \int_{-1}^{x}(1+t)dt$

$= t+\frac{t^2}{2}\Big|_{-1}^{x} = \left(x+\frac{x^2}{2}\right) - \left(-1+\frac{(-1)^2}{2}\right) = \boxed{x+\frac{x^2}{2}+\frac{1}{2}}$

Ⅲ $x\in(0,1]$: $\rightsquigarrow F(x) = \int_{-\infty}^{-1} f(t)dt + \int_{-1}^{0} f(t)dt + \int_{0}^{x} f(t)dt$

$= \int_{-\infty}^{-1} 0\, dt + \int_{-1}^{0}(1+t)dt + \int_{0}^{x}(1-t)dt = 0 + \left(t+\frac{t^2}{2}\right)\Big|_{-1}^{0} +$

$t-\frac{t^2}{2}\Big|_{0}^{x} = 0 + \left(0-\left(-1+\frac{(-1)^2}{2}\right)\right) + \left(\left(x-\frac{x^2}{2}\right)-0\right) = \boxed{-\frac{x^2}{2}+x+\frac{1}{2}}$

Ⅳ $x\in(1,\infty)$: $\rightsquigarrow F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{-1} f(t)dt$

$+\int_{-1}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{\infty} f(t)dt = 0 + \int_{-1}^{0}(1+t)dt$

$+\int_{0}^{1}(1-t)dt + 0 = t+\frac{t^2}{2}\Big|_{-1}^{0} + t-\frac{t^2}{2}\Big|_{0}^{1} = \boxed{1}$

$$\Rightarrow F(x) = \begin{cases} 0, & x<-1 \\ \frac{x^2}{2}+x+\frac{1}{2}, & -1\le x\le 0 \\ -\frac{x^2}{2}+x+\frac{1}{2}, & 0<x\le 1 \\ 1 & x>1 \end{cases}$$

③ Find $P(-\frac{1}{2}\le X\le \frac{1}{2})$ and $P(\frac{-3}{4}\le X\le 2)$

• $P(-\frac{1}{2}\le X\le\frac{1}{2}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = \int_{-\frac{1}{2}}^{0} \underbrace{f(x)}_{\to 1+X}dx + \int_{0}^{\frac{1}{2}} \underbrace{f(x)}_{1-X}dx = \int_{-\frac{1}{2}}^{0}(1+X)\,dX +$

$\int_{0}^{1/2}(1-X)dX = X+\frac{X^2}{2}\Big|_{-\frac{1}{2}}^{0} + X-\frac{X^2}{2}\Big|_{0}^{1/2} = \left[0-\left(-\frac{1}{2}+\frac{(-\frac{1}{2})^2}{2}\right)\right] + \left[\left(\frac{1}{2}-\frac{(\frac{1}{2})^2}{2}\right)-0\right]$

$= \frac{3}{8}+\frac{3}{8} = \frac{6}{8} = \boxed{\frac{3}{4}}$ ✓

• $P(\frac{-3}{4}\le X\le 2) = \int_{-\frac{3}{4}}^{2} f(x)dx = \int_{-\frac{3}{4}}^{0} \overbrace{f(x)}^{1+X}dx + \int_{0}^{1} \overbrace{f(x)}^{1-X}dx + \int_{1}^{2} \overbrace{f(x)}^{0}dx$

$= \int_{-\frac{3}{4}}^{0}(1+X)\,dX + \int_{0}^{1}(1-X)\,dX + 0 = \left(X+\frac{X^2}{2}\right)\Big|_{-\frac{3}{4}}^{0} + \left(X-\frac{X^2}{2}\right)\Big|_{0}^{1} = \left[0-\left[\frac{-3}{4}+\frac{(-\frac{3}{4})^2}{2}\right]\right]$

$+\left(\left(1-\frac{1}{2}\right)-0\right) = \frac{15}{32}+\frac{1}{2} = \boxed{\frac{31}{32}}$ ✓