Integral Calculus

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Sequences and Infinite Series

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Lecture note 27

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★ Sequences and Infinite Series ★

To understand sequences and series, you must understand how are related. The purposes of the first two sections (1. sequence & 2. Infinite Series) are to introduce sequences and series, and to illustrate both their differences and their relationships with each other.

1. Sequences :

Let's start with an example : Consider the following List of numbers:

$$\{2, 5, 8, 11, 14, \ldots\}$$

This list is an example of a sequence, where each number in the sequence is called a term of the sequence. we denote sequence in any of the following forms: $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$, $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

and note that the subscript n that appears in a_n is called an index. so, applying the above notation to our example, we have

$$a_1 = 2, \quad a_2 = 5, \quad a_3 = 8, \quad a_4 = 11, \ldots$$

as you can see, we do not have the general term a_n in this example.
but if you take a look at the terms $a_1, a_2, a_3, \ldots$, you'll find the relation between them, and you can find a pattern to guess a_n.

$$\left\{\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5, \ldots \\ 2, & 5, & 8, & 11, & 14, \ldots \end{matrix}\right\}$$

3 3 3 3

3(2)

3(3)

→ <u>we can see this sequence start with $a_1 = 2$ and then in each step we add "3" to the previous term.</u> So,

we want to start with 2

$$a_n = 2 + 3n \longrightarrow n = 0, 1, 2, \ldots$$

or

$$a_n = 2 + 3(n-1) \longrightarrow n = 1, 2, 3, \ldots$$

we want start with 2.

using this fact (underline sentences), we find out that we can write this list of number (this sequence) in the following recurrence relation formula:

$$a_n = 3 + a_{n-1},\ n = 2, 3, 4, \ldots \text{ and } a_1 = 2$$

check →

$n=2 \rightarrow a_2 = 3 + a_1 = 3 + 2 = 5$

$n=3 \rightarrow a_3 = 3 + a_2 = 3 + 5 = 8$

$n=4 \rightarrow a_4 = 3 + a_3 = 3 + 8 = 11 \ldots$

<u>Definition</u> (Sequence):

A sequence $\{a_n\}$ is an infinite ordered list of numbers of the form

$$\{a_n\} := \{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

A sequence can be defined in one of the following formula:

function that gives us the relation (pattern) between the terms $a_1, a_2, a_3, \ldots$

1. Explicit formula:

$$a_n = f(n), \qquad n = 1, 2, 3, \ldots$$

2. Recurrence relation formula:

$$a_{n+1} = f(a_n), \qquad n = 1, 2, 3, \ldots$$

each term is a function of the previous term.

Example 1 (Explicit Formula)

(I) $a_n = 2n - 1$

Here, we have the explicit formula, and try to find our sequence. we start with plugging $n=1, n=2, n=3, \ldots$ to find it.

$$\left.\begin{aligned}
n=1 &\rightarrow a_1 = 2(1)-1 = 1\\
n=2 &\rightarrow a_2 = 2(2)-1 = 3\\
n=3 &\rightarrow a_3 = 2(3)-1 = 5\\
n=4 &\rightarrow a_4 = 2(4)-1 = 7\\
&\vdots
\end{aligned}\right\} \Rightarrow \text{Thus, } \{a_n\} = \{1, 3, 5, 7, \ldots\}$$

(II) $\{3, 10, 17, 24, \ldots\}$

In this case, the sequence is given, and we need to find the explicit formula.

	a_1	a_2	a_3	a_4		
terms	3	10	17	24	...	?
n	1	2	3	4	...	n

You can see that

$$\begin{tikzcd}
3 \arrow[r, bend left, "+7"] \arrow[rr, bend right, "+2(7)"'] \arrow[rrr, bend right=60, "+3(7)"'] & 10 \arrow[r, bend left, "+7"] & 17 \arrow[r, bend left, "+7"] & 24 & \cdots
\end{tikzcd}$$

So, we can write the formula as

$a_n = 3 + 7n \quad \underline{n=0,1,2,\ldots}$

or

$a_n = 3 + 7(n-1) \quad \underline{n=1,2,3,\ldots}$

(III) $a_n = \left(\frac{1}{4}\right)^n$

$$\left.\begin{array}{l} n=1 \rightarrow a_1 = \left(\frac{1}{4}\right)^1 = \frac{1}{4} \\ n=2 \rightarrow a_2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} \\ n=3 \rightarrow a_3 = \left(\frac{1}{4}\right)^3 = \frac{1}{64} \\ n=4 \rightarrow a_4 = \left(\frac{1}{4}\right)^4 = \frac{1}{256} \\ \vdots \end{array}\right\} \Rightarrow \{a_n\} = \left\{\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \dots\right\}$$

Thus,

(IV) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$

To find a_n, it might be helpful to use the following table:

n	1	2	3	4	...	n
terms	$\frac{1}{2}$	$2/3$	$3/4$	$4/5$		? $n/_{n+1}$ ✓
pattern or guess	$\frac{1}{1+1}$	$\frac{2}{2+1}$	$\frac{3}{3+1}$	$\frac{4}{4+1}$		? $\frac{n}{n+1}$ ✓

By looking at this table, we'll see in our fraction nominator is always our "n" and the denominator is "n+1" (add +1 to nominator) !!!

So, $\boxed{a_n = \frac{n}{n+1} \quad n = 1, 2, 3, \dots,}$

(V) $a_n = \frac{(-1)^n}{n^2+1}$

$$\left.\begin{array}{l} n=1 \rightarrow a_1 = \frac{(-1)^1}{1^2+1} = \frac{-1}{2} \\ n=2 \rightarrow a_2 = \frac{(-1)^2}{2^2+1} = \frac{1}{5} \\ n=3 \rightarrow a_3 = \frac{(-1)^3}{3^2+1} = \frac{-1}{10} \\ n=4 \rightarrow a_4 = (-1)^4/4^2+1 = 1/17 \dots \end{array}\right\} \Rightarrow \{a_n\} = \left\{\frac{-1}{2}, \frac{1}{5}, \frac{-1}{10}, \frac{1}{17}, \dots\right\}$$

Thus,

(VI) $\{-1, 1, -1, 1, \dots, \}$

n	1	2	3	4	...	n
terms	-1	1	-1	1		$(-1)^n$
pattern or guess	$-1 = (-1)^1$	$(-1).(-1) = (-1)^2$	$(-1).(-1).(-1) = (-1)^3$	$(-1).(-1).(-1).(-1) = (-1)^4$	...	$\underbrace{(-1).(-1)\dots(-1)}_{n \text{ times}} = (-1)^n$

we can see this sequence alternating between $+1$ and -1. when we have odd term it is -1 and when we have even term is $+1$. so, $a_n = (-1)^n \quad n = 1, 2, \dots$

(VII) $\{0, 7, 26, 63, \dots\}$

n	1	2	3	4	...	n
terms	0	7	26	63	...	$n^3 - 1$ ✓
pattern or guess	$(1)^3 - 1$ $1 - 1$	$(2)^3 - 1$ $8 - 1$	$(3)^3 - 1$ $27 - 1$	$(4)^3 - 1$ $64 - 1$	...	$n^3 - 1$ ✓

This one is a bit hard to guess. but when the regular guess like add and subtract did not work, go for powers (like square, cube, ...) here, you can see the sequence for cube ~~a~~ is $1, 8, 27, 64, \dots$ and Now comparing with given sequence you find out they have only difference in "1". Thus, the answer would be $a_n = n^3 - 1$, $n = 1, 2, 3, 4, \dots$.

Example 2 (Recurrence Relation)

(I) $\begin{cases} a_{n+1} = 2a_n + 1 \\ a_1 = 1, \ n = 1, 2, 3, \ldots \end{cases}$

$n=1 \rightsquigarrow a_{1+1} = 2a_1 + 1 \rightsquigarrow a_2 = 2a_1 + 1 = 2(\overset{a_1}{1}) + 1 = 3$

$n=2 \rightarrow a_{2+1} = 2a_2 + 1 \rightarrow a_3 = 2a_2 + 1 = 2(\overset{a_2}{3}) + 1 = 7$

$n=3 \rightarrow a_{3+1} = 2a_3 + 1 \Rightarrow a_4 = 2a_3 + 1 = 2(\underset{a_3}{7}) + 1 = 15$

$\vdots$

Thus,

$\{a_n\} = \{3, 7, 15, \ldots\}$

$\begin{cases} a_{n+1} = 2a_n + 1 \\ a_1 = -1, \ n = 1, 2, 3, \ldots \end{cases}$

$n=1 \rightarrow a_2 = 2a_1 + 1 = 2(\overset{a_1}{-1}) + 1 = -1$

$n=2 \rightarrow a_3 = 2a_2 + 1 = 2(\overset{a_2}{-1}) + 1 = -1$

$n=3 \rightarrow a_4 = 2a_3 + 1 = 2(\overset{a_3}{-1}) + 1 = -1$

$\vdots$

$\{a_n\} = \{-1, -1, -1, \ldots\} = \{-1\}$

(II) $\begin{cases} a_{n+1} = \dfrac{a_n}{a_n - 1} + n \\ a_1 = 4 \end{cases} \quad n \geq 1$

$n=1 \rightarrow a_{1+1} = \dfrac{a_1}{a_1 - 1} + 1 \Rightarrow a_2 = \dfrac{a_1}{a_1 - 1} + 1 = \dfrac{4}{4-1} + 1 = \dfrac{4}{3} + 1 = \dfrac{7}{3}$

$n=2 \rightarrow a_{2+1} = \dfrac{a_2}{a_2 - 1} + 2 \Rightarrow a_3 = \dfrac{a_2}{a_2 - 1} + 2 = \dfrac{7/3}{7/3 - 1} + 2 = \dfrac{7/3}{4/3} + 2 = \dfrac{15}{4}$

$n=3 \rightarrow a_{3+1} = \dfrac{a_3}{a_3 - 1} + 3 \Rightarrow a_4 = \dfrac{a_3}{a_3 - 1} + 3 = \dfrac{11/4}{\frac{11}{4} - 1} + 3 = \dfrac{11/4}{\frac{7}{4}} + 3 = \dfrac{32}{7}$

$\vdots \qquad \vdots \qquad \vdots$

$\{a_n\} = \left\{ \dfrac{7}{3}, \dfrac{15}{4}, \dfrac{32}{7}, \ldots \right\}$

③

1.2 Limit of a Sequence

The most important question about a sequence is this:
If you go farther and farther out in the sequence how do the terms of the sequence behave? Do they approach a specific numbers (converges) or they grow in magnitude without bound (diverges)?
To find a limit of a sequence we take $n \to \infty$, and comput $\lim_{n\to\infty} a_n$. if it converges $\rightsquigarrow$ sequence converges . . –
if it diverges $\rightarrow$ sequence diverges . . .

Definition (limit of a sequence)

Let $\{a_n\}$ be a sequence and $L \in \mathbb{R}$. we say that the $\{a_n\}$ Converges to L if $\{a_n\}$ is sufficiently close to L for all n large enough. In this case, we say

$\lim_{n\to\infty} a_n = L$ $\rightsquigarrow$ "the limit of a_n is L, as n approaches infinity"

otherwise, if the sequence $\{a_n\}$ has no limit, we say that the $\{a_n\}$ Diverges.

Summary:

$\lim_{n\to\infty} a_n = L$ exist $\rightarrow$ Converges

$\lim_{n\to\infty} a_n =$ doesn't exist $\rightarrow$ Diverges

Limit Laws:

★ Theorem (limits of sequence from limits of functions)

Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L.

Example. Consider $a_n = \frac{1}{n^2+1}$. So, $f(n) = \frac{1}{n^2+1} = a_n$. Now, replace "n" by "x". So, $f(x) = \frac{1}{x^2+1}$. Taking $\lim_{x\to\infty} \frac{1}{x^2+1} = \frac{1}{\infty} = 0$, we get $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$.

★ Theorem (limit Laws for sequences)

Assume that the sequence $\{a_n\}$ and $\{b_n\}$ have limit A and B, respectively. Then,

1. $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$
2. $\lim_{n\to\infty} ca_n = cA$, where c is a real number
3. $\lim_{n\to\infty} a_n b_n = A.B$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, provided $B \neq 0$.

Example 3. (limit of a Sequence)

Determine the limit of the following sequences. Converges or Diverges !?

(I) $a_n = \frac{1}{n}$

$\frac{1}{\text{very large}} = 0$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \rightsquigarrow \{a_n\} \text{ converges to "0"}$$

(II) $a_n = \dfrac{4n^5}{12n^5+n^3-1}$

$\frac{\infty}{\infty} \rightsquigarrow$ we need to do some algebra...

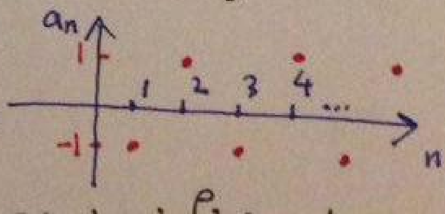
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n^5}{12n^5+n^3-1} = \lim_{n\to\infty} \frac{\frac{4\cancel{n^5}}{\cancel{n^5}}}{\frac{12n^5+n^3-1}{n^5}} = \lim_{n\to\infty} \frac{4}{12\frac{n^5}{n^5}+\frac{n^3}{n^5}-\frac{1}{n^5}}$$

$$= \lim_{n\to\infty} \frac{4}{12+\cancel{\frac{1}{n^2}}^{0}-\cancel{\frac{1}{n^5}}^{0}} = \frac{4}{12} = \frac{1}{3}. \text{ Thus, } \boxed{\lim_{n\to\infty} a_n = \frac{1}{3}} \text{ Converges to } \frac{1}{3}$$

(III) $a_n = n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty \rightarrow \{a_n\} = \{n\} \quad \text{diverges}$$

(IV) $a_n = (-1)^n \rightsquigarrow \{-1, 1, -1, 1, -1, 1, \dots\} \rightarrow$



This sequence alternating between -1 and $+1$, so, in infinity also we can not find out which value between $+1$ and -1 is attained by this sequence. So, this $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n$ does not exist and this sequence diverges.

(V) $a_n = \dfrac{(-1)^n n}{n^2+1} \rightsquigarrow \left\{ \underset{n=1}{\overset{a_1}{\dfrac{-1}{2}}}, \underset{n=2}{\overset{a_2}{\dfrac{2}{5}}}, \underset{n=3}{\overset{a_3}{\dfrac{-3}{10}}}, \underset{n=4}{\overset{a_4}{\dfrac{4}{17}}}, \dots \right\}$

The terms decrease in magnitude and approach zero with alternating signs. So, the limit appears to be 0. $\left(\lim\limits_{n\to\infty} \dfrac{(-1)^n n}{n^2+1} = 0\right)$.

In other words, if you consider $\dfrac{n}{n^2+1}$ and $(-1)^n$ separately.

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n/n}{n^2/n + 1/n} = \lim_{n\to\infty} \frac{1}{n+\cancelto{0}{\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n} = 0$$

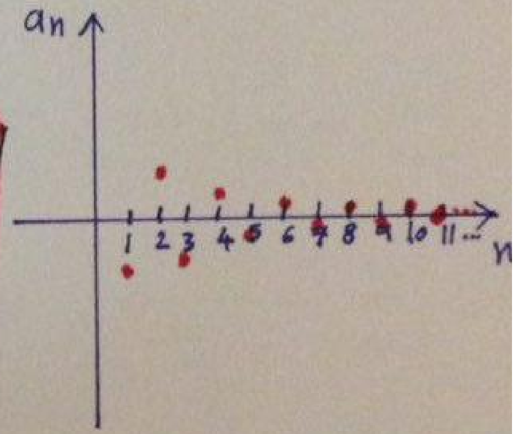
but $(-1)^n$ alternating signs. Now as $n\to\infty$, $(-1)^n$ still alternating between $+1$ and -1, but $\dfrac{n}{n^2+1} \to 0$. therefore, $(\pm 1).0 = 0$, as $n\to\infty$. (See the following Figure)

(VI) $a_n = \dfrac{2e^n+1}{e^n}$ $\quad \dfrac{\infty}{\infty} \to$ same algebra

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2e^n+1}{e^n} = \lim_{n\to\infty} \frac{\frac{2e^n+1}{e^n}}{\frac{e^n}{e^n}}$$

$$= \lim_{n\to\infty} \frac{2\frac{e^n}{e^n} + \frac{1}{e^n}}{\frac{e^n}{e^n}} = \lim_{n\to\infty} \frac{2+\cancelto{0}{\frac{1}{e^n}}}{1} = 2$$

Note that $\lim\limits_{n\to\infty} \dfrac{1}{e^n} = 0$ (since $\dfrac{1}{e^\infty} = \dfrac{1}{\infty} = 0$). $\{a_n\}$ converges to 2



(VII) $a_n = \dfrac{n^2 - n^3 + 1}{7n^2+n-5}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-n^3+1}{7n^2+n-5} = \lim_{n\to\infty} \frac{\frac{-n^3+n^2+1}{n^2}}{\frac{7n^2+n-5}{n^2}} = \lim_{n\to\infty} \frac{-n+1+\cancelto{0}{\frac{1}{n^2}}}{7+\cancelto{0}{\frac{1}{n}}-\cancelto{0}{\frac{5}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{-n+1}{7} = -\infty \to \{a_n\} \text{ diverges to } \underline{\underline{-\infty}}.$$

Example 4 (a bit harder):

(I) $a_n = \sqrt{n^2+1} - n$. To find $\lim_{n\to\infty} a_n$, we need to multiply the conjugate of $\sqrt{n^2+1} - n$, which is $\sqrt{n^2+1} + n$ in nominator and denominator.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n^2+1} - n \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n}$$

$$= \frac{1}{\text{large number}_{\to\infty}} = 0$$!!! $\{a_n\}$ converges to "0".

(II) $a_n = \left(\frac{n+5}{n}\right)^n$

See Example 1 (page 608)
Textbook

(III) $a_n = n^{1/n}$

See Example 1 (page 608)
Textbook

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Example 5 Compare the $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$, where

$$a_n = \frac{n^{5/2}+1}{2n^{5/2}-3} \quad \text{and} \quad b_n = \frac{(-1)^n(n^{5/2}+1)}{2n^{5/2}-3} = (-1)^n a_n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^{5/2}+1}{2n^{5/2}-3}$$

$$= \lim_{n\to\infty} \frac{\frac{n^{5/2}+1}{n^{5/2}}}{\frac{2n^{5/2}-3}{n^{5/2}}} = \lim_{n\to\infty} \frac{1+\cancel{\frac{1}{n^{5/2}}}^{\,0}}{2-\cancel{\frac{3}{n^{5/2}}}^{\,0}}$$

$$= \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}.$$

$\{a_n\}$ converges to $\frac{1}{2}$

Using the result of part (a), it follows that even terms approach $\frac{1}{2}$ and odd terms approach $-\frac{1}{2}$. Therefore, the sequence diverges, illustrating the fact that the presence of $(-1)^n$ <u>may</u> significantly alter the behavior of a sequence.

■ Squeeze Theorem

★ Theorem (Squeeze theorem for sequences)

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \le b_n \le c_n$ for all integers n greater than some index N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Summary

$\lim_{n\to\infty} a_n = L$ $\qquad$ $\lim_{n\to\infty} c_n = L$

$$a_n \le b_n \le c_n \longrightarrow \lim_{n\to\infty} b_n = L$$

$\searrow$ L $\qquad$ L

Example 6. (Squeeze Theorem) Find the limit of the sequences $a_n = \frac{\sin n}{n^2+1}$, $b_n = \frac{(-1)^n}{n}$, and $c_n = \frac{\cos(\frac{n\pi}{2})}{\sqrt{n}}$

- $a_n = \frac{\sin n}{n^2+1}$, use the fact that $-1 \le \sin n \le 1$. Thus, $-\frac{1}{n^2+1} \le a_n \le \frac{1}{n^2+1}$.

In otherwords, $-\frac{1}{n^2+1} \le \frac{\sin n}{n^2+1} \le \frac{1}{n^2+1}$. Then, since $\lim\limits_{n\to\infty} \frac{1}{n^2+1} = 0$ and $\lim\limits_{n\to\infty} -\frac{1}{n^2+1} = 0$, by squeeze theorem, we get $\lim\limits_{n\to\infty} \frac{\sin n}{n^2+1} = 0$

- $b_n = \frac{(-1)^n}{n} \rightsquigarrow \frac{-1}{n} \le b_n = \frac{(-1)^n}{n} \le \frac{1}{n}$. $\lim\limits_{n\to\infty} \frac{1}{n} = \lim\limits_{n\to\infty} \frac{-1}{n} = 0$

So, by squeeze theorem $\rightsquigarrow \lim\limits_{n\to\infty} \frac{(-1)^n}{n} = 0$

- $c_n = \frac{\cos(\frac{n\pi}{2})}{\sqrt{n}}$ $\rightsquigarrow$ Note that $\cos(\frac{n\pi}{2})$ for $n=1,2,3,\dots$ is always $0, 1$ or -1 $\rightarrow$ $\frac{-1}{\sqrt{n}} \le \frac{\cos(\frac{n\pi}{2})}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$

$\rightsquigarrow \lim\limits_{n\to\infty} \frac{-1}{\sqrt{n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Thus, by squeeze Theorem $\lim\limits_{n\to\infty} c_n = 0$

$\cos(\frac{n\pi}{2})$:
$\cos(\pi/2) = 0$ $\cos(\frac{3\pi}{2}) = 0$ ...
$\cos(\pi) = -1$ $\cos(2\pi) = 1$

★ Terminology for Sequence ★

We now introduce some terminology for sequences that is similar to that used for functions.

- $\{a_n\}$ is increasing if $a_{n+1} > a_n$; Example, $\{1, 2, 3, 4, \dots\}$
- $\{a_n\}$ is nondecreasing if $a_{n+1} \ge a_n$; Example, $\{1, 1, 1, 2, 2, 3, 3, \dots\}$
- $\{a_n\}$ is decreasing if $a_{n+1} < a_n$; Example, $\{10, 9, 8, 7, \dots\}$
- $\{a_n\}$ is nondecreasing if $a_{n+1} \le a_n$; Example, $\{0, -1, -1, -2, -2, \dots\}$
- $\{a_n\}$ is monotonic if it is either nonincreasing or non-decreasing (it moves in one direction)
- $\{a_n\}$ is bounded if there is number M such that $|a_n| \le M$, for all relevant values of n.

⑥ Geometric Sequences

Geometric sequences have the property that each term is obtained by multiplying the previous term by a fixed constant, called the ratio. They have the following form

$\{r^n\}$ or $\{ar^n\}$ ⤳ the ratio "r"

$a \neq 0$ ⤳ real number

example ⤳ • $\left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \cdots\right\}$ ⤳ $\begin{cases} r = \frac{1}{2} \\ a = 1 \end{cases}$

or $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots\right\}$

• $\left\{\frac{-4}{5}, \frac{-4}{5^2}, \frac{-4}{5^3}, \frac{-4}{5^4}, \cdots\right\}$ ⤳ $\begin{cases} r = \frac{1}{5} \\ a = -4 \end{cases}$

Example 7: (Geometric Sequences) Find the limit of the following sequences.

(I) $a_n = \left(\frac{3}{4}\right)^n$ $\lim_{n\to\infty} \frac{3^n}{4^n} = \frac{\infty}{\infty}$. but between 3^n and 4^n 4^n goes faster to ∞. Therefore, $\lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0$

(II) $b_n = \left(\frac{-3}{4}\right)^n$

With the same argument in example 7, we get

$$\lim_{b\to\infty} b_n = \lim_{n\to\infty} (-1)^n \left(\frac{3}{4}\right)^n = 0.$$

(III) $\left(\frac{8}{5}\right)^n = c_n$

$c_n = \left(\frac{8}{5}\right)^n = \frac{8^n}{5^n} \rightsquigarrow \lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{8^n}{5^n} = \frac{\infty}{\infty}$. Note that 8^n goes faster to ∞ than 5^n. Thus, at infinity, 8^n wins and is stronger. Then, $\lim_{n\to\infty} c_n = \lim \left(\frac{8}{5}\right)^n = \infty \rightsquigarrow \{c_n\}$ diverges

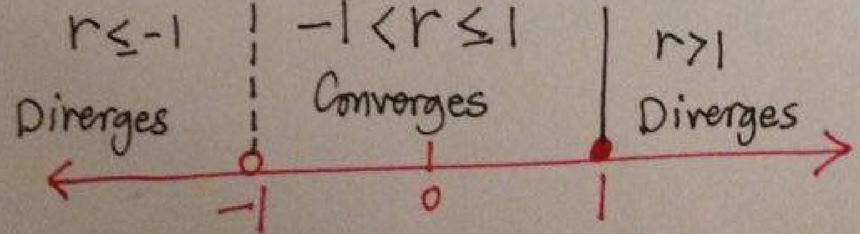
(IV) $d_n = \left(\frac{-8}{5}\right)^n$

$d_n = \left(\frac{-8}{5}\right)^n = (-1)^n \left(\frac{8}{5}\right)^n$. Since $\lim_{n\to\infty} \left(\frac{8}{5}\right)^n = \infty$ and $(-1)^n$ alternating between $+1$ and -1, $d_n = \left(\frac{-8}{5}\right)^n$ diverges.

★ Theorem (Geometric Sequence) ★

Let "r" be a real number. Then,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & -1 < r < 1 \\ 1 & r = 1 \\ \text{does not exist} & r > 1 \text{ or } r \le -1 \end{cases}$$



Also, $\begin{cases} r > 0 \rightsquigarrow \{r^n\} \text{ monotonic} \\ r < 0 \rightsquigarrow \{r^n\} \text{ oscillates} \end{cases}$

Example 8: Find the limit of the following sequences:

(I) $a_n = \left(\frac{2}{9}\right)^n$

In this sequence $r = \frac{2}{9}$. since $-1 < r = \frac{2}{9} < 1$, we have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\frac{2}{9}\right)^n = 0$

(II) $b_n = \left(\frac{-10}{7}\right)^n$

Here, $r = \frac{-10}{7}$. since $-1 > r = \frac{-10}{7}$, we have $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\left(\frac{-10}{7}\right)^n \to$ diverges

(III) $c_n = \frac{2^n + 5^n}{6^n}$ ⇝ we can rewrite it as $c_n = \frac{2^n}{6^n} + \frac{5^n}{6^n} = \left(\frac{1}{3}\right)^n + \left(\frac{5}{6}\right)^n$.

- For $\left(\frac{1}{3}\right)^n \rightsquigarrow -1 < r = \frac{1}{3} < 1 \rightsquigarrow \lim\limits_{n\to\infty}\left(\frac{1}{3}\right)^n = 0$
- For $\left(\frac{5}{6}\right)^n \to -1 < r = \frac{5}{6} < 1 \rightsquigarrow \lim\limits_{n\to\infty}\left(\frac{5}{6}\right)^n = 0$

$\Rightarrow \lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} \frac{2^n + 5^n}{6^n} = 0$

(V) $d_n = \frac{4^n - 8^n}{9^n}$ ⇝ we can rewrite it as $d_n = \frac{4^n}{9^n} - \frac{8^n}{9^n} = \left(\frac{4}{9}\right)^n - \left(\frac{8}{9}\right)^n$.

- For $\left(\frac{4}{9}\right)^n \rightsquigarrow -1 < r = \frac{4}{9} < 1 \rightsquigarrow \lim\limits_{n\to\infty}\left(\frac{4}{9}\right)^n = 0$
- For $\left(\frac{8}{9}\right)^n \rightsquigarrow -1 < r = \frac{8}{9} < 1 \rightsquigarrow \lim\limits_{n\to\infty}\left(\frac{8}{9}\right)^n = 0$

$\Rightarrow \lim\limits_{n\to\infty} d_n = \lim\limits_{n\to\infty} \frac{4^n - 8^n}{9^n} = 0$

(IV) $e_n = \left\{2^{n+1} 3^{-n}\right\}$ ⇝ we can rewrite it as $e_n = \frac{2^{n+1}}{3^n} = \frac{2^n \cdot 2}{3^n} = 2\left(\frac{2}{3}\right)^n$.

- For $\left(\frac{2}{3}\right)^n \rightsquigarrow -1 < r = \frac{2}{3} < 1 \rightsquigarrow \lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$. Thus, $\lim\limits_{n\to\infty} e_n = \lim\limits_{n\to\infty} 2\left(\frac{2}{3}\right)^n = 2 \lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$

2 0

(VI) $k_n = \left\{5(\quad)^n\right\}$

since $r = -1.01$, and $-1.01 < -1$, we have $\lim\limits_{n\to\infty}(-1.01)^n = 0$. Thus, $\lim\limits_{n\to\infty} k_n = 0$

(VII) $p_n = \left\{\frac{2^n + \cos n}{5^n}\right\}$ ⇝ we can rewrite it as $p_n = \frac{2^n}{5^n} + \frac{\cos n}{5^n} = \left(\frac{2}{5}\right)^n + \frac{\cos n}{5^n}$

- For $\left(\frac{2}{5}\right)^n \to -1 < r = \frac{2}{5} < 1 \rightsquigarrow \lim\limits_{n\to\infty}\left(\frac{2}{5}\right)^n = 0$
- For $\frac{\cos n}{5^n}$, we use squeeze theorem, since $-1 \leq \cos n \leq 1$ we have $\frac{-1}{5^n} \leq \frac{\cos n}{5^n} \leq \frac{1}{5^n}$. We have also $\lim\limits_{n\to\infty} \frac{-1}{5^n} = \lim\limits_{n\to\infty} \frac{1}{5^n} = 0$, by squeeze theorem ⇝ $\lim\limits_{n\to\infty} \frac{\cos n}{5^n} = 0$

$\Rightarrow$ Thus, $\lim\limits_{n\to\infty} p_n = \lim\limits_{n\to\infty} \frac{2^n + \cos n}{5^n} = 0$