Integral Calculus

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Function of 2 variables

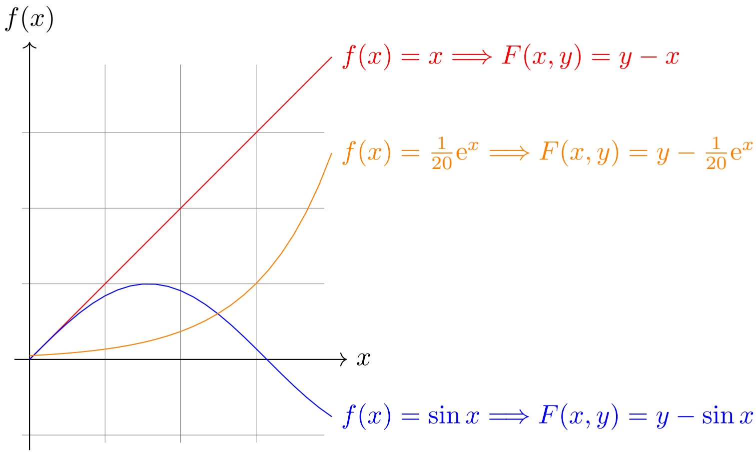
0.1 Recall on a function of one variable

A variable is a quantity that can take a range of values (e.g. population, poverty rate, etc.). A function of one variable takes a variable x and produces another variable y. The explicit form of the function is:

$$y = f(x)$$

Here x is an **independent** variable and y is a **dependent** variable. It means that the value of y (or $f(x)$) depends on that of x. Equations in two variables can be written out of the function (it is its implicit form):

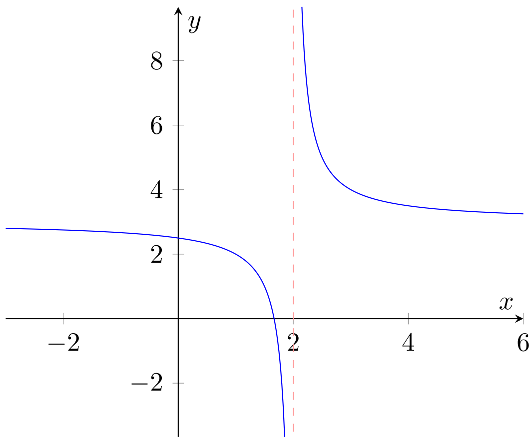
$$F(x, y) = 0$$



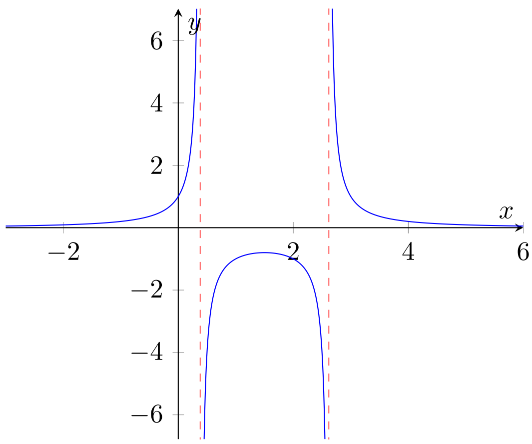
Important: Domain and range of a function

The **domain** $\mathcal{D}_f$ of a function $y = f(x)$ is a subset of $\mathbb{R}$ consisting of all points x that are allowed to be considered as inputs for the function $f(x)$. In other words, all the points for which $f(x)$ is defined.
The **range** of f is the set of real number y that the function $f(x)$ can take.

Example 0.1. $y = f(x) = \dfrac{1}{x-2}$. The function $f(x)$ is not defined if $x - 2 = 0$. It means that x should not be equal to 2. So the domain of $f(x)$ is $\mathcal{D}_f = \{x \in \mathbb{R} | x \neq 2\}$ or $\mathcal{D}_f =]-\infty; 2[\cup]2; +\infty[$



Example 0.2. $y = f(x) = \dfrac{1}{x^2 - 3x + 1}$. The function $f(x)$ is not defined for $x^2 - 3x + 1 = 0$. For that we have to find the roots of $x^2 - 3x - 1 = 0$. So we have $x_1 = \dfrac{3-\sqrt{5}}{2}$ and $x_2 = \dfrac{3+\sqrt{5}}{2}$. This means that the function $f(x)$ is discontinuous at these two points. So, we have $\mathcal{D}_f = \left\{ x \in \mathbb{R} | x \neq \dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2} \right\}$ or $\mathcal{D}_f = \left] -\infty; \dfrac{3-\sqrt{5}}{2} \right[\cup \left] \dfrac{3+\sqrt{5}}{2}; +\infty \right[$



0.2 Function of two variables (sec. 12.2)

A function of two variables takes two variables x and y and produces another variable z. The explicit form is written as follows:

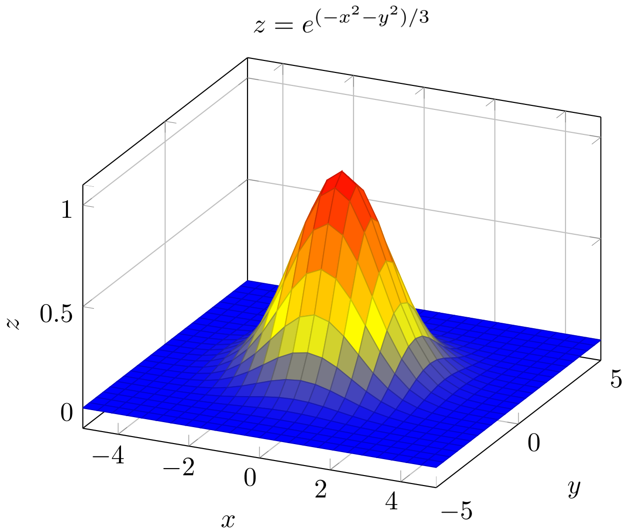
$$z = f(x, y)$$

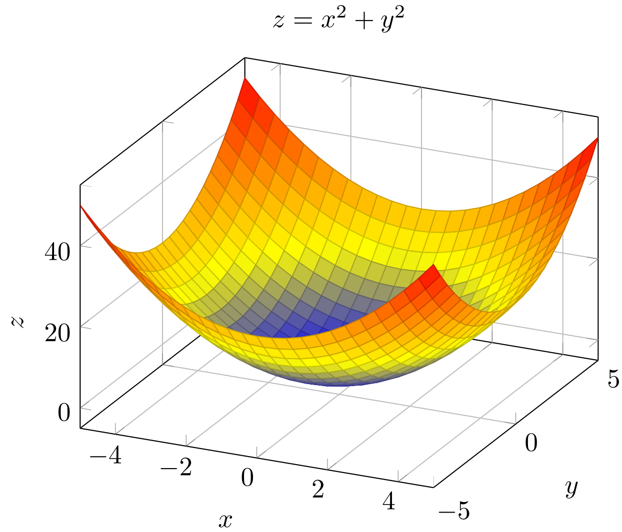
For each point (x, y) a unique value of z is assigned by the function $z = f(x, y)$. In other words, the input is the point (x, y) and the output is z. Here the **independent** variables are x and y. Whereas the depend variable is z.

The implicit form is written as follows:

$$F(x, y) = 0$$

Example 0.3. $z = f(x, y) = e^{(-x^2-y^2)/3} \implies F(x, y, z) = z - e^{(-x^2-y^2)/3} = 0$
$z = f(x, y) = x^2 + y^2 \implies F(x, y, z) = z - x^2 - y^2 = 0$





Domain and range with two independent variables

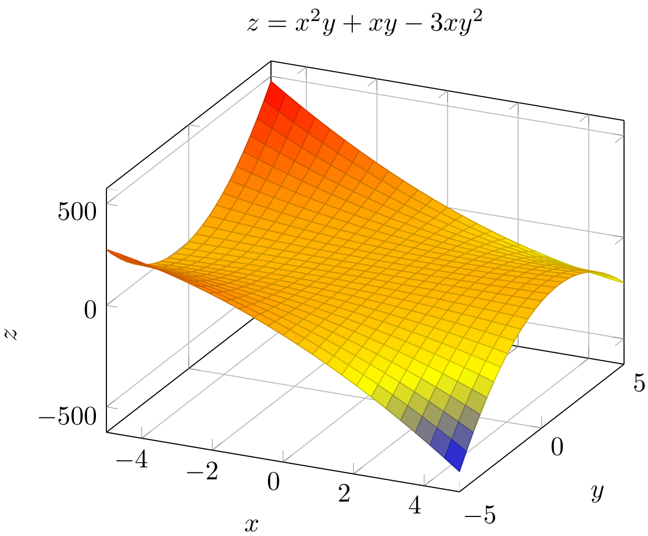
Definition 1. **Domain**. The set of (x, y) points, called $\mathcal{D}_f$, on the $xy-$plane for which the function $f(x, y)$ is defined.

Definition 2. **Range**. It is the set of real numbers z that the function $f(x, y)$ can take.

Related Exercises sec. 12.2 11–20
In general the function of two variables $f(x, y)$ can be written in the following form:

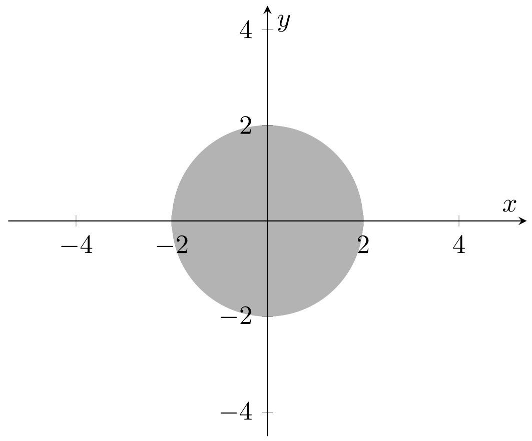
- **Polynomial** function: $f(x, y) = U(x, y)$. $\mathcal{D}_f = \{(x, y) \in \mathbb{R}^2\}$

  Example 0.4. $z = f(x, y) = x^2y + xy - 3xy^2$. Here the domain is defined as: $\mathcal{D}_f = \{(x, y) \in \mathbb{R}^2\}$ because we can pick up all (x, y) points on the $xy-$plane.



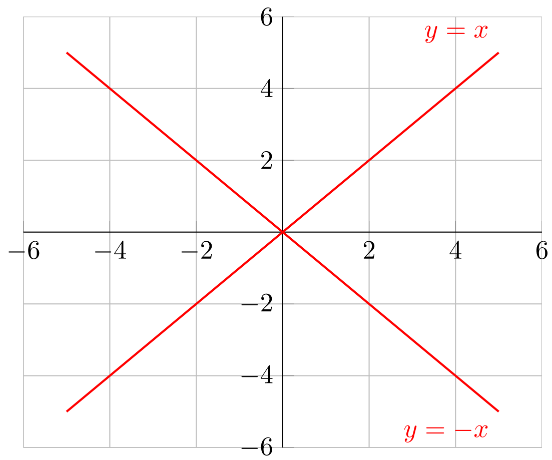
- **Irrational** function: $f(x,y) = \sqrt{U(x,y)}$. Here, the function $f(x,y)$ is **not defined** if $U(x,y) < 0$. It means that we can not have negative numbers under the square root. So $\mathcal{D}_f = \{(x,y) \in \mathbb{R}^2 | U(x,y) \geqslant 0\}$

  Example 0.5. $f(x,y) = \sqrt{4 - x^2 - y^2}$. Here, f is defined for $4 - x^2 - y^2 \geqslant 0$ or $x^2 + y^2 \leqslant 4$. It means that the domain is all (x,y) points corresponding to $x^2 + y^2 \leqslant 4$. This is an equation of a circle of radius $r = \sqrt{4} = 2$ centered at $(0,0)$. Here $\mathcal{D}_f = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leqslant 4\}$.



- **Rational** function: $f(x,y) = \dfrac{U(x,y)}{V(x,y)}$. A rational form is defined if the denominator is not equal to 0 $(V(x,y) \neq 0)$

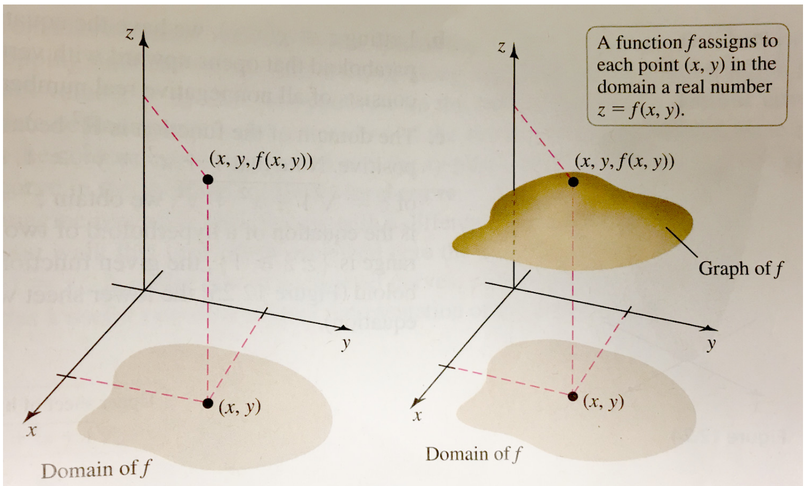
  Example 0.6. $f(x,y) = \dfrac{x}{x^2 - y^2}$. Here $f(x,y)$ is defined if and only $x^2 - y^2 \neq 0$, which means that $x \neq \pm y$. So the domain of the function f is $\mathcal{D}_f = \{(x,y) \in \mathbb{R}^2 | x \neq \pm y\}$. Here we can pick up any point (x,y) expect those on the $y = x$ and $y = -x$ curves.



Graph, trace and level curves of a function of two variables

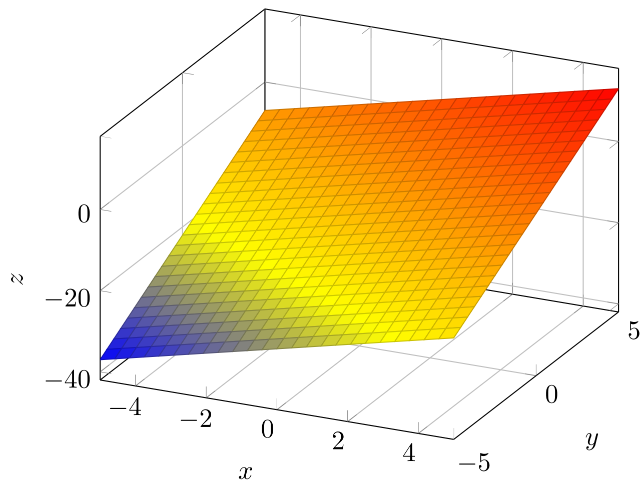
0.3 Graph of a function of two variables

Definition 3. Graph. It is the set of points (x, y, z) that satisfy the equation $z = f(x, y)$. More specifically, for each point (x, y) on the $xy-$plane in the domain of f, the point (x, y, z) lies on the **graph** of f.



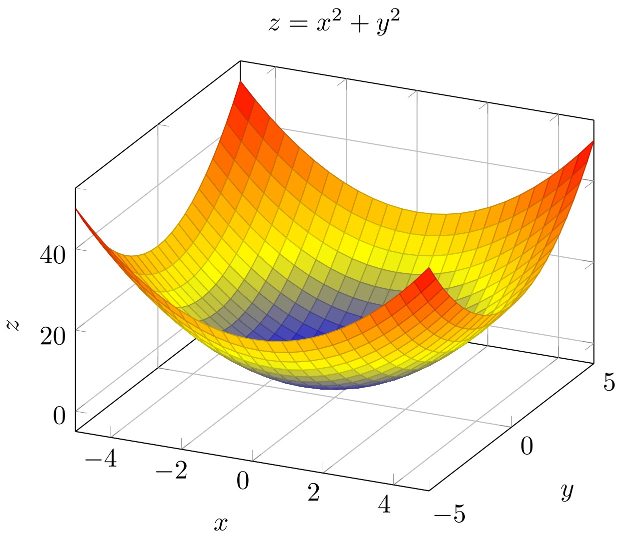
Related Exercises sec 12.2 21–29

Example 0.7. Graphing two-variable functions. Find the domain and range of $z = f(x, y) = 2x+3y-12$. Here f is defined for all (x, y) points in $\mathbb{R}^2$ and the range is all the z-value in $\mathbb{R}$. So $\mathcal{D}_f = \{(x, y) \in \mathbb{R}^2\}$.



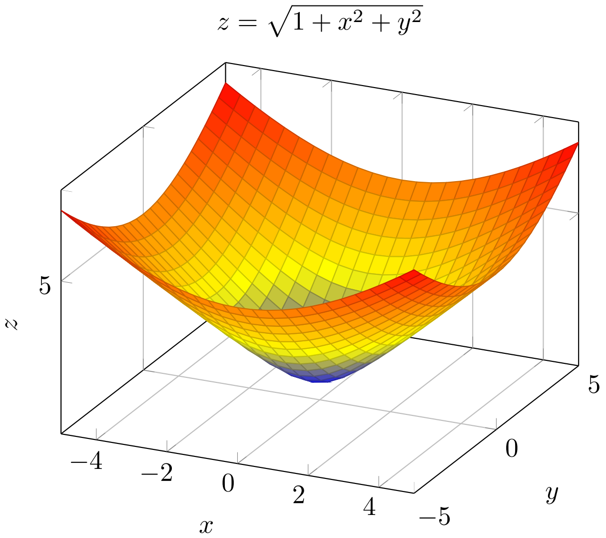
Example 0.8. Graphing two-variable functions. Find the domain and range of $z = f(x, y) = x^2 + y^2$. Here f is defined for all (x, y) points in $\mathbb{R}^2$ and the range is all nonnegative z-value in $\mathbb{R}$. So $\mathcal{D}_f = \{(x, y) \in \mathbb{R}^2\}$.

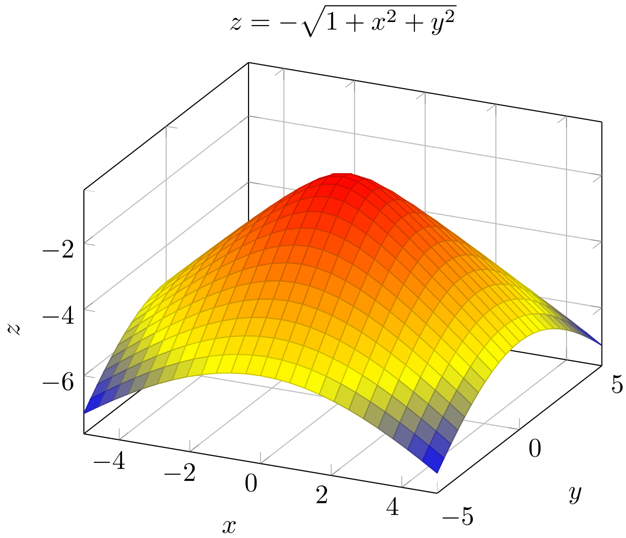
We have a paraboloid $z = x^2 + y^2$.



Example 0.9. Graphing two-variable functions. Find the domain and range of $z = f(x, y) = \sqrt{1 + x^2 + y^2}$. Here f is defined for $1 + x^2 + y^2 \geqslant 0$. Note that $1 + x^2 + y^2 \geqslant 1$ so the range is $\{z : z \geqslant 1\}$. Squaring both sides of the $f(x, y)$, we have $z^2 = 1 + x^2 + y^2$ or $-x^2 - y^2 + z^2 = 1$. This is the equation of a hyperboloid of two sheets that opens along the $z-$axis.

So the domain of definition is $\mathcal{D}_f = \{(x, y) \in \mathbb{R}^2\}$.



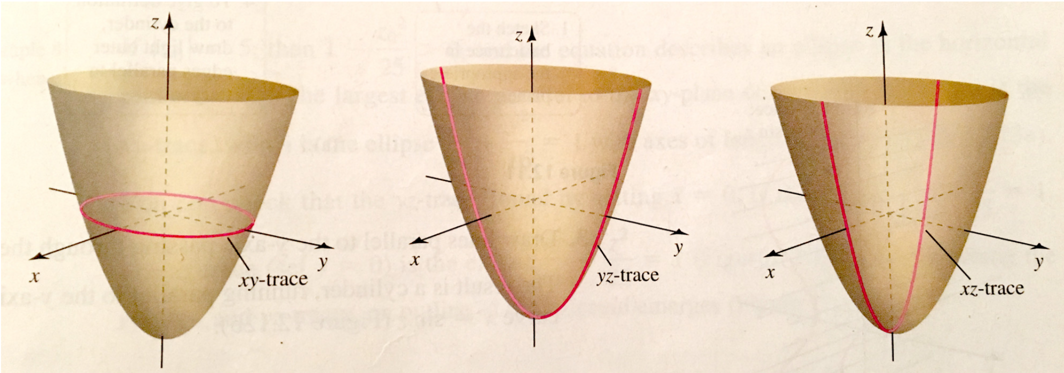


These are the typical graphs that we are going to see in this section:

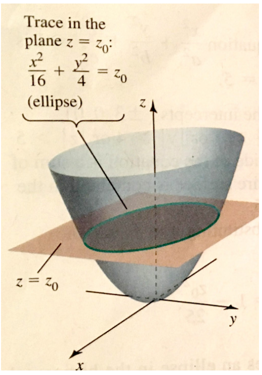
- Sphere centered at $(0, 0, 0)$ with a radius a: $x^2 + y^2 + z^2 = a^2$
- Elliptic Paraboloid: $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$, $a \neq b$
- Elliptic Hyperboloid: $-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

0.4 Traces of a function of two variables (sec. 12.1)

Definition 4. **Trace.** It is a set of points at which the surface **intersects** a plane that is that is **parallel** to one of the **coordinate planes**. The traces in the coordinate planes are called $xy-$**trace**, the $yz-$**trace** and the $xz-$**trace**. *Related Exercises sec. 12.1 51–54*



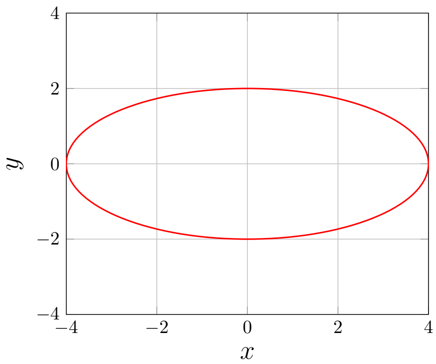
Example 0.10. Given a graph $z = \dfrac{x^2}{16} + \dfrac{y^2}{4}$. Find and sketch the trace in $z = 1$.



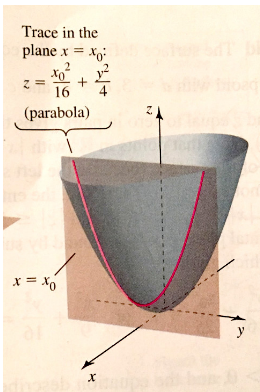
Intersection at $z = 1 \Longrightarrow 1 = \dfrac{x^2}{16} + \dfrac{y^2}{4}$. So we have

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

This is the equation of an ellipse (center $(0,0,0)$, $a = 4$ and $b = 2$)



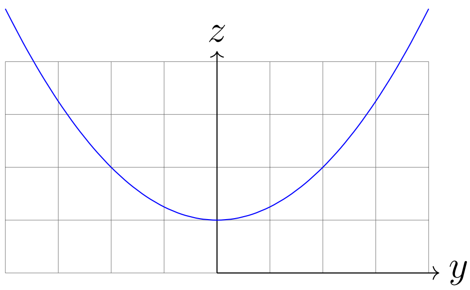
Example 0.11. Given a graph $z = \dfrac{x^2}{16} + \dfrac{y^2}{4}$. Find and sketch the trace in $x = -4$.



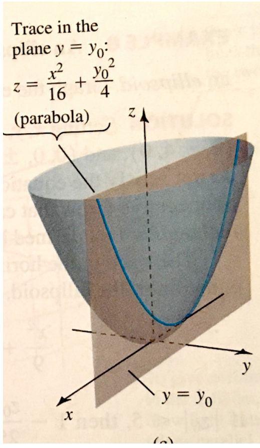
Intersection at $x = -4 \Longrightarrow z = \dfrac{(-4)^2}{16} + \dfrac{y^2}{4}$. So we have

$$z = \frac{y^2}{4} + 1$$

This is a *parabola*



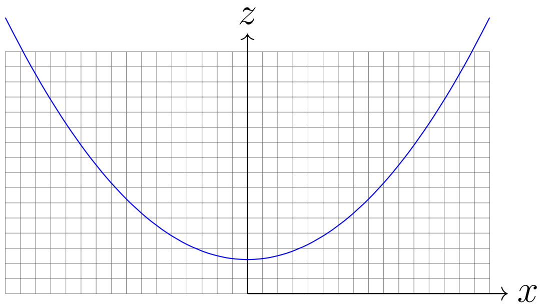
Example 0.12. Given a graph $z = \dfrac{x^2}{16} + \dfrac{y^2}{4}$. Find and sketch the trace in $y = 3$.



Intersection at $y = 3 \Longrightarrow z = \dfrac{x^2}{16} + \dfrac{(3)^2}{4}$. So we have

$$z = \frac{x^2}{16} + \frac{9}{4}$$

This is a *parabola*



0.5 Level Curves

Functions of two variables presented by surfaces in $\mathbb{R}^3$. Instead of a 3D view, we are flattening everything in 2D plane. It is used in topography for instance.

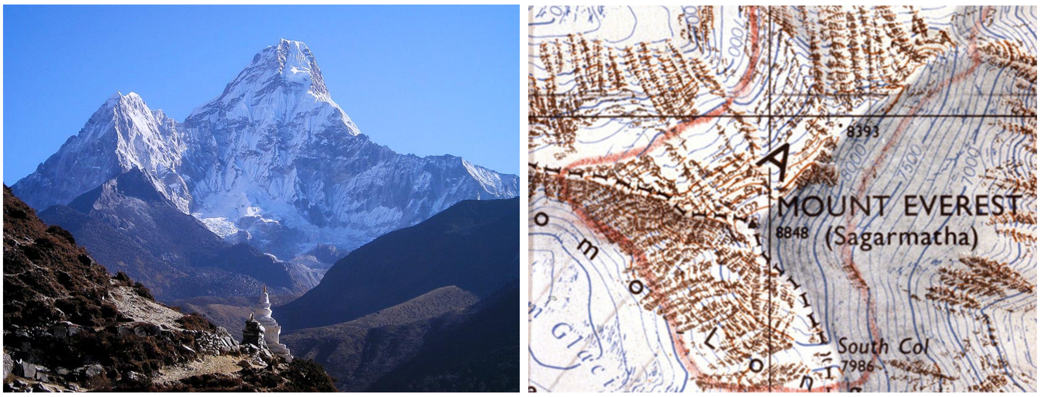
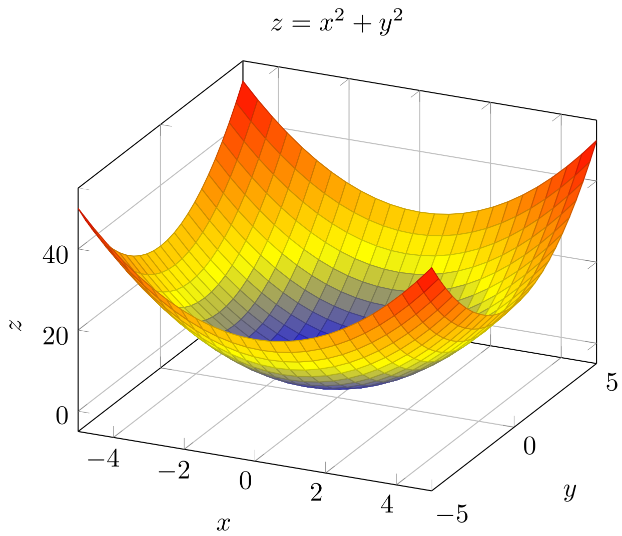
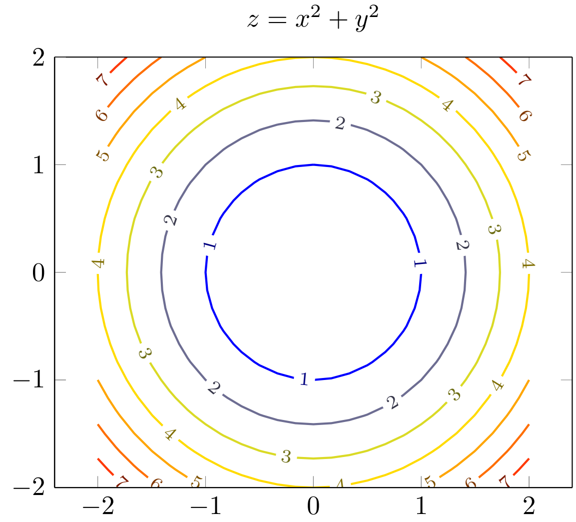


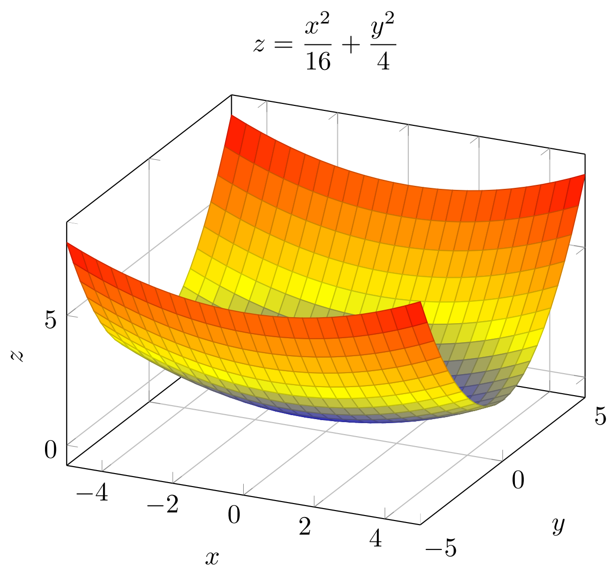
Figure 1: Map of Mount Everest using level curves (traces).

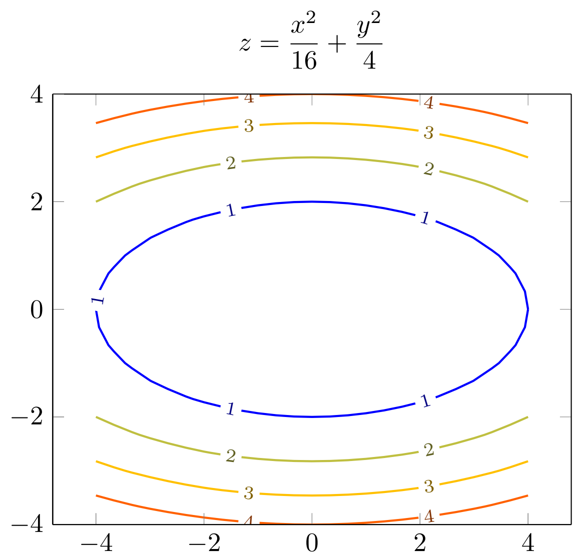
Example 0.13. $z = \dfrac{x^2}{4} + \dfrac{y^2}{4}$. Find the levels for $z = 1, 2, 3, 4, 5, 6, 7$. We just have to replace the value of z in the equation and find the corresponding ellipse equations.
We know how to find the equations of traces and sketch them. So we have,





Example 0.14. $z = \dfrac{x^2}{16} + \dfrac{y^2}{4}$. Find the levels for $z = 1, 2, 3, 4$. We just have to replace the value of z in the equation and find the corresponding ellipse equations.
We know how to find the equations of traces and sketch them. So we have,





Remarks

- The equation of a circle of radius r and centered at $(0, 0)$ is $x^2 + y^2 = r^2$
- The equation of an ellipse centered at $(0, 0)$ is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$