Integral Calculus

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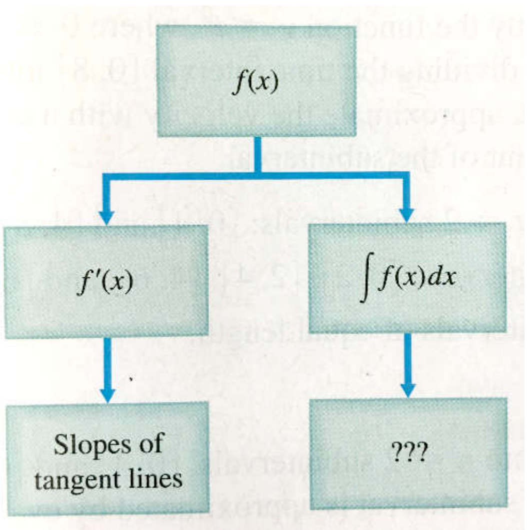
1 Approximating Area under Curves (textbook section 5)



1.1 Area under a curve

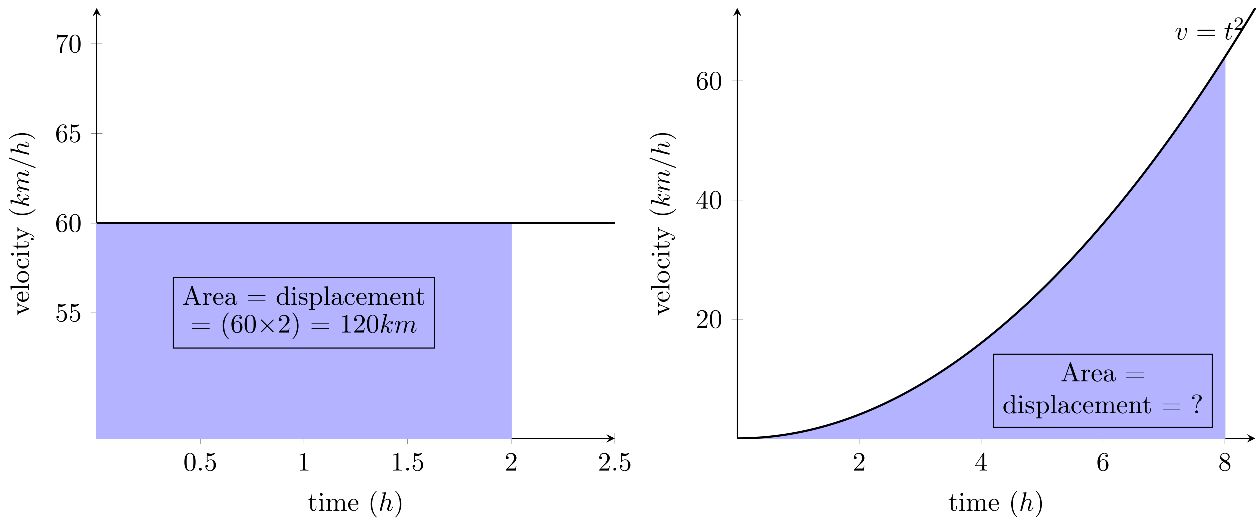


Figure 1: Displacement of a car at a constant velocity $60km/h$ and a car moving at a velocity given by $v = t^2$.

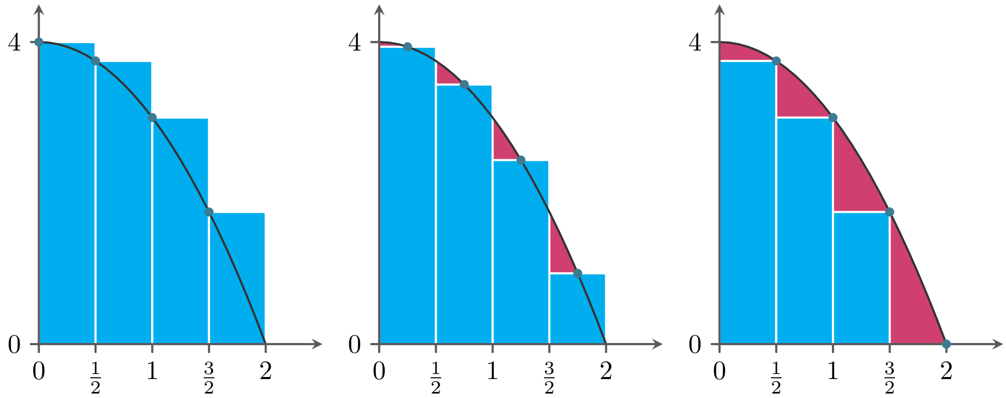
1.2 How to calculate the area?

1. Approximate the area using rectangles

2. Better approximation to get smaller error

Approximation using rectangles

Here we want to approximate the area under the curve for which $x \in [a, b]$ where $a = 0$ and $b = 2$. Three cases can be considered:



- **Case 1:** Left end points

  Here the interval $x \in [0, 2]$ is divided in 4 sub-interval whose lengths are all equal (here, spaced by $\Delta x = 0.5$). The **upper-left** end of the rectangles are intersecting the function. The sum of the rectangles gives an **approximation** of the area under the curve.
  Area $= f(0) \cdot \Delta x + f(1/2) \cdot \Delta x + f(1) \cdot \Delta x + f(3/2) \cdot \Delta x$

- **Case 2:** Mid points

  Here the interval $x \in [0, 2]$ is divided in 4 sub-interval whose lengths are all equal (here, spaced by $\Delta x = 0.5$). The function is intersecting the rectangles at their **upper-mid** edges. The sum of the rectangles gives an **approximation** of the area under the curve.
  Area $= f(1/4) \cdot \Delta x + f(3/4) \cdot \Delta x + f(5/4) \cdot \Delta x + f(7/4) \cdot \Delta x$

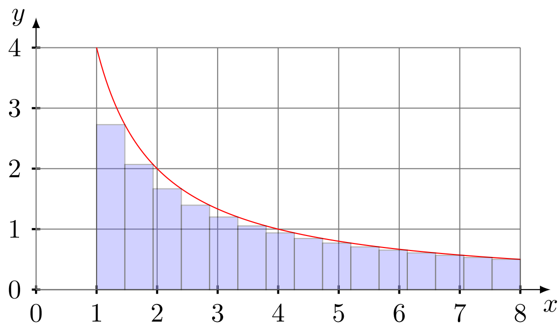
- **Case 3:** Right end points

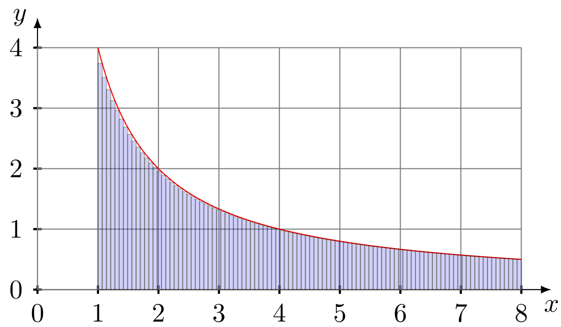
  Here the interval $x \in [0, 2]$ is divided in 3 sub-interval whose lengths are all equal (here, spaced by $\Delta x = 0.5$). The **upper-right** end of the rectangles are intersecting the function. The sum of the rectangles gives an **approximation** of the area under the curve.
  Area $= f(1/2) \cdot \Delta x + f(1) \cdot \Delta x + f(3/2) \cdot \Delta x$

How to decrease the error?

Better approximation means more rectangles. In other words, the interval $[a, b]$ should be divided into more subintervals. If we decrease the thickness of the rectangles, we have smaller rectangles because the area of each rectangle is defined by $f(x_i) \cdot \Delta x$ where Δx is given by $\Delta x = \dfrac{b-a}{n}$ (n is the number of subintervals). The

Figure 2: credit: www.indiana.edu/∼rcapub/v22n2/p19.html





area under the curve is then given by the sum of all of these rectangle and written as follows:

$$Area = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

This sum is called **Riemann Sum** for f on $[a, b]$:

- a **left Riemann sum** if we use the **left end point** (R_n)
- a **right Riemann sum** if we use the **right end point** (L_n)
- a **midpoint Riemann sum** if the use the **midpoint** (M_n)

Definition 1. Regular position. Suppose $[a, b]$ is a closed interval containing n subintervals

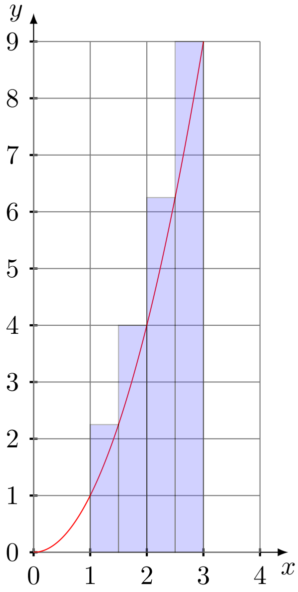
$$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$$

of equal length $\Delta x = \dfrac{b-a}{n}$ with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, \cdots, x_{n-1}, x_n$ of the subintervals are called **grid points**, and they create a **regular position** of the interval $[a, b]$. In general, the ith grid point is:

$$x_i = a + i\Delta x, \qquad \text{for } i = 0, 1, 2, \cdots, n.$$

Example 1.1. Left and right Riemann sums. Let R be the region bounded by the graph of $f(x) = x^2$ between $x = 1$ and $x = 3$. Estimate the area using 4 approximating rectangles.

- **Right endpoint** (R_4)

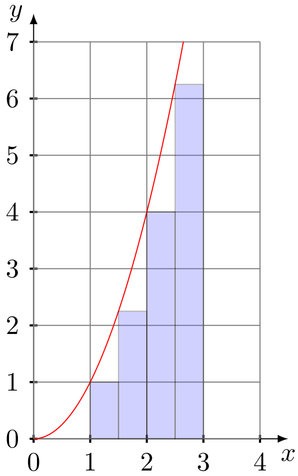


$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$$

R_4 is the sum of all the areas using right endpoints

$$R_4 = f(3/2)\frac{1}{2} + f(2)\frac{1}{2} + f(5/2)\frac{1}{2} + f(3)\frac{1}{2}$$
$$R_4 = (3/2)^2\frac{1}{2} + (2)^2\frac{1}{2} + (5/2)^2\frac{1}{2} + (3)^2\frac{1}{2}$$
$$R_4 = \frac{43}{4}$$

- **Left endpoint** (L_4)

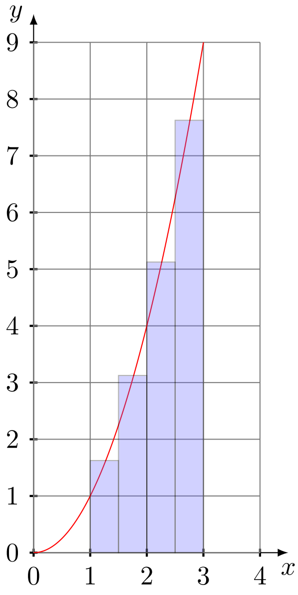


$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$$

L_4 is the sum of all the areas using left endpoints

$$L_4 = f(1)\frac{1}{2} + f(3/2)\frac{1}{2} + f(2)\frac{1}{2} + f(5/2)\frac{1}{2}$$
$$L_4 = (1)^2\frac{1}{2} + (3/2)^2\frac{1}{2} + (2)^2\frac{1}{2} + (5/2)^2\frac{1}{2}$$
$$L_4 = \frac{27}{4}$$

- **Midpoint** (M_4)



$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$$

M_4 is the sum of all the areas using midpoints

$$M_4 = f(5/4)\frac{1}{2} + f(7/4)\frac{1}{2} + f(9/4)\frac{1}{2} + f(11/4)\frac{1}{2}$$
$$M_4 = (5/4)^2\frac{1}{2} + (7/4)^2\frac{1}{2} + (9/4)^2\frac{1}{2} + (11/4)^2\frac{1}{2}$$
$$M_4 = \frac{69}{16}$$

Working with Riemann sum is cumbersome with large number of subintervals. So to avoid writing this $Area = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x + f(x_7)\Delta x + f(x_8)\Delta x + f(x_9)\Delta x + \cdots + f(x_{n-1})\Delta x + f(x_n)\Delta x$, there is a way to write it in more compact form.

$$Area = \sum_{i=1}^{n} f(x_i)\Delta x$$

Example 1.2.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = \sum_{i=1}^{7} i = 28$$

$$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 = \sum_{i=0}^{7} i = 28$$

$$\sum_{i=0}^{4}(i+1) = (0+1) + (1+1) + (2+1) + (3+1) + (4+1) = 1 + 2 + 3 + 4 + 5 = 15$$

Some properties

Suppose that $\{a_1, a_2, a_3, \cdots, a_n\}$ and $\{b_1, b_2, b_3, \cdots, b_n\}$ are sets of real numbers and c a real number. Then,

- $\sum\limits_{i=0}^{n} ca_i = c\sum\limits_{i=0}^{n} a_i$
- $\sum\limits_{i=0}^{n}(a_i + b_i) = \sum\limits_{i=0}^{n} a_i + \sum\limits_{i=0}^{n} b_i$
- $\sum\limits_{i=0}^{n}(a_i - b_i) = \sum\limits_{i=0}^{n} a_i - \sum\limits_{i=0}^{n} b_i$

Some useful sums

Let n be a positive integer and c a real number. Then,

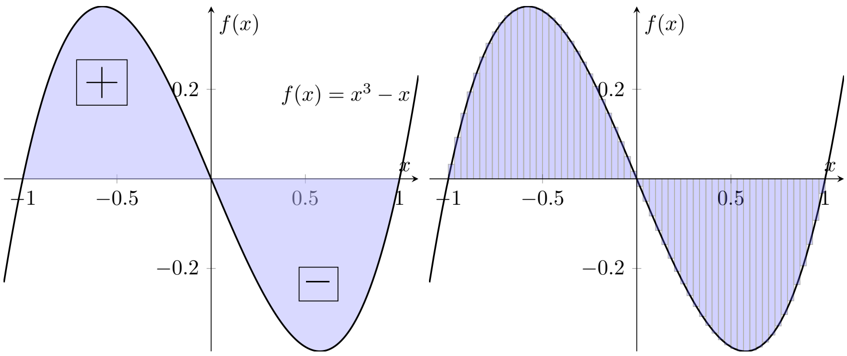
- $\sum\limits_{i=1}^{n} c = cn$
- $\sum\limits_{i=1}^{n} i = \dfrac{n(n+1)}{2}$
- $\sum\limits_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$
- $\sum\limits_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

2 Definite Integral

2.1 Net Area

Definition 2. Net Area. Let $\mathcal{R}$ be bounded by the graph of the continuous function f and the $x-$axis between $x = a$ and $x = b$. So, the **Net Area** is given by:

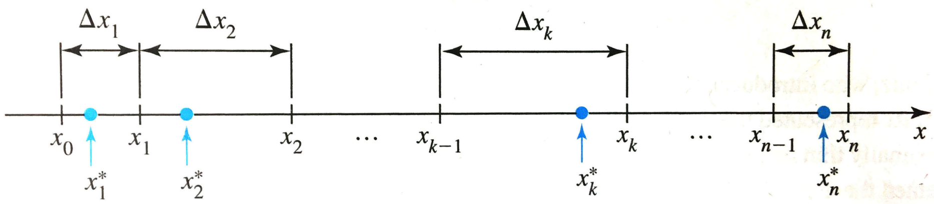
$$\text{Net Area} = \text{The sum of the areas above the } x\text{-axis} - \text{The sum of the areas below the } x\text{-axis}$$



Definition 3. Generalised Riemann Sum. Suppose that $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$ of $[a, b]$ with:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Let Δx_k the length of the subinterval $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, \cdots, n$



If f is defined on $[a, b]$, the sum:

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

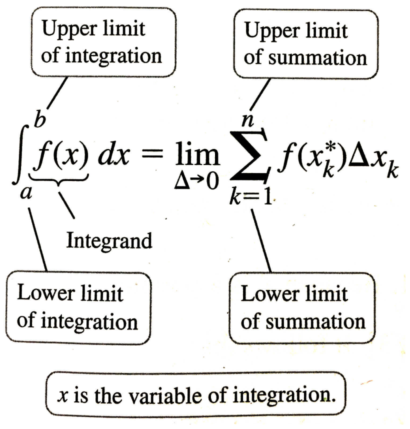
is called a **general Riemann sum for** f **on** $[a, b]$.

2.2 Definite Integral

Definition 4. A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim\limits_{\Delta x \to 0} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x_i$ exists and is unique over all positions of $[a, b]$ and all choices of x_i^* on a partition. This limit is the **definite integral of** f **from** a **to** b, which we write:

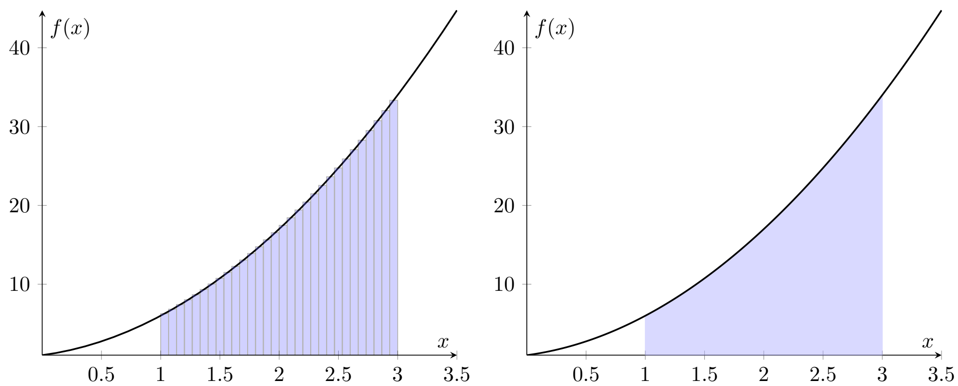
$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Terminology



Example 2.1. Assume that $\lim\limits_{\Delta x\to 0} \sum\limits_{k=1}^{n}(3{x_k}^{*2} + 2x_k^* + 1)\Delta x_k$ is the limit of a Riemann sum for a function f on $[1, 3]$. Identify the function f and express the limit as a defined integral. What does the integral represent geometrically?
solution:

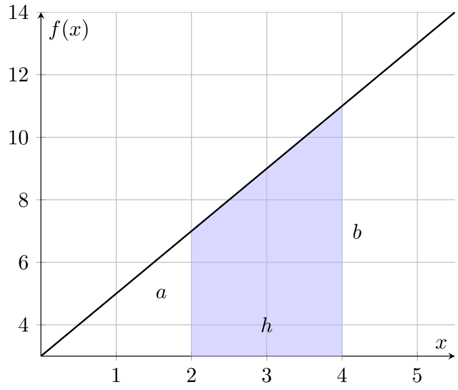
$$\lim_{\Delta x\to 0} \sum_{k=1}^{3}(3{x_k}^{*2} + 2x_k^* + 1)\Delta x_k = \int_1^3 (3x^2 + 2x + 1)dx$$



Example 2.2. Evaluating definite integrals using geometry. Use familiar area formulation to evaluate the following definite integral:

$$\int_2^4 (2x+3)dx$$

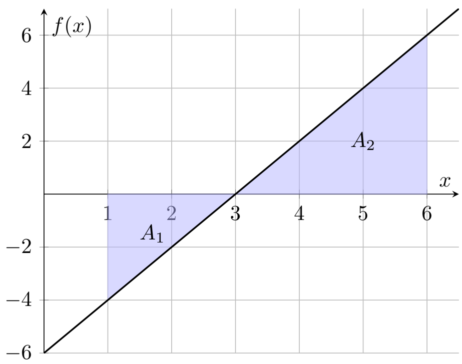
Here we have a trapezoid, its area is $A = \frac{1}{2}h(a+b) = \frac{1}{2}2(7+11) = 18$



Example 2.3. Evaluating definite integrals using geometry. Use familiar area formulation to evaluate the following definite integral:

$$\int_1^6 (2x-6)dx$$

Here 2 triangles, the total area is $A = A_1 + A_2$, where $A_1 = \frac{2\times 4}{2} = 4$ and $A_2 = \frac{3\times 6}{2} = 9$.

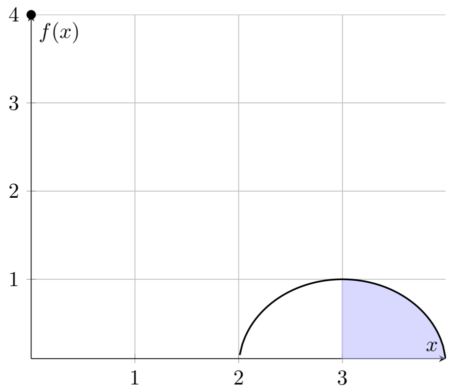


So $A = 9 - 4 = 5$

Example 2.4. Evaluating definite integrals using geometry. Use familiar area formulation to evaluate the following definite integral:

$$\int_3^4 \sqrt{1-(x-3)^2}dx$$

Here we have a quarter of a disk. Its area is $A = \dfrac{1}{4}\pi r^2 = \dfrac{\pi}{4}$



Related Exercises sec. 5.1 25–32

2.3 Properties of Definite Integrals

Let f and g be integrable functions on an interval that contains a, b and p.

1. $\int_a^a f(x)dx = 0$ *Definition*
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ *Definition*
3. $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
4. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ *for any constant c*
5. $\int_a^b f(x)dx = \int_a^p f(x)dx + \int_p^b f(x)dx$
6. The function $|f|$ is integrable on $[a,b]$, and $\int_a^b |f(x)|dx$ is the sum of the areas of the regions bounded by the graph of f and the $x-$axis on $[a,b]$

Related Exercises sec. 5.2 41–46

2.4 Evaluating Definite Integrals Using Limits

Given a definite integral $\int_a^b f(x)dx = \lim\limits_{\Delta x\to 0}\sum\limits_{i=1}^{n} f(x_i^*)\Delta x_i$.

To express it as a limit of a sum, we compute $\Delta x = \dfrac{b-a}{n}$ (here Δx does not change) and we know that $x_i^* = x_i = a + i\Delta x$ for a Right Riemann Sum (to simplify the calculations). We know, as well, that $\Delta x \to 0$ when $n \to \infty$. So the evaluation of the integral can be written as follows:

$$\int_a^b f(x)dx = \lim_{\Delta x\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\cdot\frac{b-a}{n}$$

Example 2.5. Express the following integral as a limit of Riemann Sum:

$$\int_1^3 \ln(x)dx$$

Solution: First, Δx is given by $\Delta x = \frac{3-1}{n} = \frac{2}{n}$. Then $x_i^* = x_i = a + i\Delta x = 1 + i\frac{2}{n}$. Applying the previous definition, we express the integral as a Riemann Sum:

$$\int_1^3 \ln(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \ln(1 + i\frac{2}{n}) \cdot \frac{2}{n}$$

Example 2.6. Evaluating definite integral using limits. Find the value of the following function by evaluating a Right Riemann sum and letting $n \to 0$.

$$\int_0^2 (x^3 + 1)dx$$

Solution: First, Δx is given by $\Delta x = \frac{2-0}{n} = \frac{2}{n}$. Then $x_i^* = x_i = a + i\Delta x = 0 + i\frac{2}{n}$. Applying the previous definition, we express the integral as a Riemann Sum:

$$\begin{aligned}\int_0^2 (x^3 + 1)dx = \lim_{n\to\infty} \sum_{i=1}^{n} ((i\frac{2}{n})^3 + 1) \cdot \frac{2}{n} &= \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{8i^3}{n^3} + 1) \cdot \frac{2}{n} \\ &= \lim_{n\to\infty} \left(\frac{8}{n^3} \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} 1 \right) \cdot \frac{2}{n} \\ &= \lim_{n\to\infty} \left(\frac{8}{n^3} \frac{n^2(n+1)^2}{4} + n \right) \cdot \frac{2}{n} \\ &= \lim_{n\to\infty} \left(4\frac{n^2 + 2n + 1}{n^2} + 2 \right)\end{aligned}$$

$$\int_0^2 (x^3 + 1)dx = 6$$