Integral Calculus

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1 Fundamental theorem of Calculus

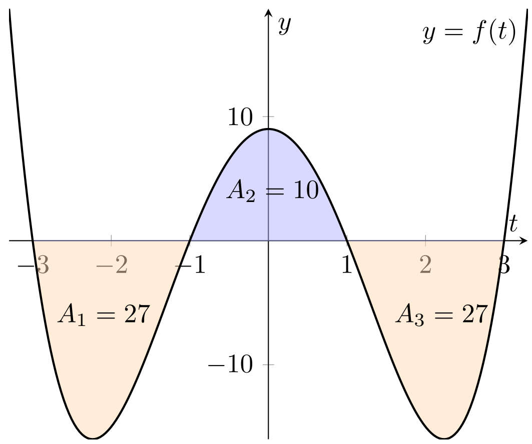
1.1 Area function and definite integrals

Definition 1. Let f be continuous function, for $t \geqslant a$. The **area function for** f **with left endpoint** a is:

$$A(x) = \int_a^x f(t)dt$$

where $x \geqslant a$. The area function gives the net area of the region bounded by the graph of f and the $t-$axis on the interval $[a, x]$.

Example 1.1. Comparing area functions. The following graph of f has areas at various regions. Let $A(x) = \int_{-3}^x f(t)dt$ and $F(x) = \int_{-1}^x f(t)dt$ be two area functions for f.



Evaluate the following area functions:

1. $A(-1)$ and $F(-1)$
2. $A(1)$ and $F(1)$
3. $A(3)$ and $F(3)$

Solution:

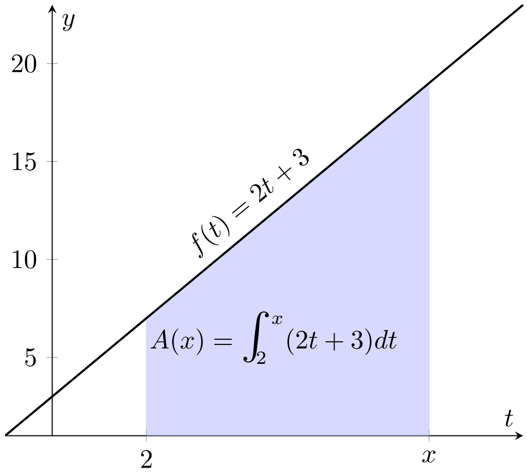
1. The value of $A(-1) = \int_{-3}^{-1} f(t)dt$ is the net area of the region bounded by the graph of f and the $t-$axis on the interval $[-3, -1]$. We see $A_1 = 27$. So $A(-1) = -27$

   On the over hand, $F(-1) = \int_{-1}^{-1} f(t)dt = 0$ (*Property 1*). Notice that $A(-1) - F(-1) = -27$

2. The value of $A(1) = \int_{-3}^{1} f(t)dt$ is found by subtracting the area below the $t-$axis on $[-3,-1]$ and the area above the $t-$axis on $[-1,1]$. Therefore, we have $A(1) = 10 - 27 = -17$. Similarly, $F(1)$ is the net area of the region bounded by the graph of f and the $t-$axis on $[-1,1]$. Therefore, $F(1) = 10$. Notice that $A(1) - F(1) = -27$.

3. Reasoning as in part (1) and (2), we see that $A(3) = -27 + 10 - 27 = -44$ and $F(3) = 10 - 27 = -17$. As before observe that $A(3) - F(3) = -27$

Example 1.2. Area of a trapezoid. Consider the trapezoid bounded by $f(t) = 2t + 3$ and the $t-$axis from $t = 2$ to $t = x$. $A(x) = \int_{2}^{x} (2t+3)dt$ gives the area of the trapezoid, for $x \geqslant 2$



1. Evaluate $A(2)$

2. Evaluate $A(5)$

3. Find and graph the area function $y = A(x)$, for $x \geqslant 2$

4. Compare the derivative of A to f.

Solution:

1. By Property 1 $A(2) = \int_{2}^{2} (2t+3)dt = 0$

2. $A(5)$ is the area of the trapezoid on the interval $[2,5]$. Using the area formula of a trapezoid, we have:

$$A(5) = \int_{2}^{5} (2t+3)dt = \frac{1}{2}(5-2)\cdot(f(2)+f(5)) = \frac{1}{2}\cdot 3(7+13) = 30$$

3. Now the endpoint is the variable $x \geqslant 2$. The distance between the parallel sides is $x - 2$.

$$A(x) = \frac{1}{2}(x-2)\cdot(f(2)+f(x)) = \frac{1}{2}(x-2)(7+2x+3)$$

$$A(x) = \int_{2}^{x} (2t+3)dt = x^2 + 3x - 10$$

4. Differentiating the area function:

$$A'(x) = \frac{d}{dx}(x^2 + 3x - 10) = 2x + 3 = f(x)$$

Therefore $A'(x) = f(x)$, or equivalently, the area function A is an antiderivative of f. **It is the first part of the Fundamental Theorem of Calculus**

First part of the Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ then the area function $A(x) = \int_a^x f(t)dt$, for $a \leqslant x \leqslant b$ is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies:

$$A'(x) = \frac{d}{dx}A(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Which means that **the area function of f is an antiderivative of f on** $[a, b]$.

Relation between $A(x)$ and the antiderivative of f

We know that area function $A(x)$ is an antiderivative of f. Let $F(x)$ be any other antiderivative of $f(x)$ on $[a, b]$. The relation between $A(x)$ and $F(x)$ is:

$$A(x) = F(x) + c, \qquad \text{for } a \leqslant x \leqslant b$$

Noting that $A(a) = 0$, it follows that:

$$A(b) - A(a) = A(b) = [F(b) + c] - [F(a) + c]$$

Thus,

$$A(b) = \int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

It is the second part of the Fundamental Theorem of Calculus

Second part of the Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$A(b) = \int_a^b f(x)dx = F(b) - F(a)$$

Example 1.3. Evaluating definite integrals. Evaluate the following definite integrals using the Fundamental Theorem of Calculus.

$$\int_0^2 (x^2 + 1)dx; \qquad \int_0^{\frac{\pi}{2}} 3\sin(x)dx; \qquad \int_1^2 (\sqrt{t} - t^2)dx$$

Solutions:

- $\int_0^2 (x^2 + 1)dx$

$$\int_0^2 (x^2 + 1)dx = \int_0^2 x^2 dx + \int_0^2 1dx = \frac{x^3}{3}\Big|_0^2 + x\Big|_0^2$$
$$= \left[\frac{2^3}{3} - \frac{0^3}{3}\right] + [2 - 0] = \frac{8}{3} + 2 = \frac{14}{3}$$

- $\int_0^{\frac{\pi}{2}} 3\sin(x)dx$

$$\int_0^{\frac{\pi}{2}} 3\sin(x)dx = 3\int_0^{\frac{\pi}{2}} \sin(x)dx = -3\cos x\Big|_0^{\frac{\pi}{2}} = -3[\cos(\frac{\pi}{2}) - \cos(0)] = -3(0) + 3(1) = 3$$

- $\int_0^2 (\sqrt{t} - t^2)dx$

$$\int_1^2 (\sqrt{t} - t^2)dt = \int_1^2 \sqrt{t}dt + \int_1^2 t^2dt = \frac{2}{3}t^{3/2}\Big|_1^2 + \frac{1}{3}t^3\Big|_1^2 = \frac{2}{3}[2^{3/2} - 1^{3/2}] - \frac{1}{3}[2^3 - 1^3]$$
$$= \frac{2}{3}\sqrt{8} - 3$$

Related Exercises sec. 5.3 51–60

1.2 Derivative of Integrals

Using the **Part 1 of the Fundamental Theorem of Calculus**, we can compute the derivative of an integral. (Note that we should use also the chain rule !!!).
In this section, the goal is to get $f(x)$

Example 1.4. Simplify the following expressions:

$$\frac{d}{dx}\int_1^x (\sqrt{t} - t^2)dt = \sqrt{x} - x^2$$
$$\frac{d}{dx}\int_x^1 3\sin(t)dt = -\frac{d}{dx}\int_1^x 3\sin(t)dt = -3\sin x$$
$$\frac{d}{dx}\int_2^{x^3} \ln(t^2)dt = 3x^2\ln(x^6)$$

General formula for the derivative of an integral

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$$

Related Exercises sec. 5.3 61–68