Homework week 3 solutions

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1 Implicit differentiation

The function $z = f(x, y)$ obeys:

$$f(x, y) + \sin(f(x, y)) = 2xy(x + 1)$$

1. Find $\frac{\partial f}{\partial x}(0, 0)$ **(2 points)**
2. Find $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$ **(2 points)**

Solution

1. Find $\frac{\partial f}{\partial x}(0, 0)$

   Applying $\frac{\partial}{\partial x}$ to $f(x, y) + \sin(f(x, y)) = 2xy(x + 1)$ gives:

   $$\frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial x}(x, y)\cos(f(x, y)) = 2y(2x + 1) \qquad \textbf{(1 point)}$$

   Setting $x = y = 0$, we have:

   $$\frac{\partial f}{\partial x}(0, 0) + \frac{\partial f}{\partial x}(0, 0)\cos(f(0, 0)) = \frac{\partial f}{\partial x}(0, 0)\,(1 + \cos(f(0, 0)) = 0$$

   Here we have $\frac{\partial f}{\partial x}(0, 0) = 0$ **(1 point)**

2. Find $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$

   Applying $\frac{\partial}{\partial y}$ to $\frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial x}(x, y)\cos(f(x, y)) = 2y(2x + 1)$ gives:

   $$\frac{\partial^2 f}{\partial x \partial y}(x, y) + \frac{\partial^2 f}{\partial x \partial y}(x, y)\cos(f(x, y)) - \frac{\partial f}{\partial x}(x, y)\frac{\partial f}{\partial y}(x, y)\sin(f(x, y)) = 2(2x + 1) \qquad \textbf{(1 point)}$$

   Setting $x = y = 0$, we have:

   $$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y}(0, 0) + \frac{\partial^2 f}{\partial x \partial y}(0, 0)\cos(f(0, 0)) - \frac{\partial f}{\partial x}(0, 0)\frac{\partial f}{\partial y}(0, 0)\sin(f(0, 0)) &= 2 \\ \frac{\partial^2 f}{\partial x \partial y}(0, 0)\,(1 + \cos(f(0, 0))) &= 2 \end{aligned}$$

   We have

   $$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{2}{1 + \cos(f(0, 0))} \qquad \textbf{(1 point)}$$

2 Analysing critical points

Use the second derivative test to classify the critical points of $f(x, y) = xy(x-1)(y+2)$. **(6 points)**

Solution

First derivatives:

$$f_x = (2x-1)(y^2+2y) = 0 \quad \textbf{(0.5 point)} \tag{1}$$

$$f_y = (x^2-x)(2y+2) = 0 \quad \textbf{(0.5 point)} \tag{2}$$

Eq. (1) gives: $y = 0$ or $y = -2$ or $x = 1/2$.

- for $y = 0$, Eq. (2) gives $x = 0$ or $x = 1 \Longrightarrow$ critical points $(0, 0)$ and $(1, 0)$ **(0.5 point)**
- for $y = -2$, Eq. (2) gives $x = 0$ or $x = 1 \Longrightarrow$ critical points $(0, -2)$ and $(1, -2)$ **(0.5 point)**
- for $x = 1/2$, Eq. (2) gives $y = -1 \Longrightarrow$ critical point $(1/2, 1)$ **(0.5 point)**

Second Derivative Test:

$f_{xx} = 2(y^2 + 2y)$; **(0.5 point)** $\quad f_{yy} = 2(x^2 - x)$; **(0.5 point)** $\quad f_{xy} = (2x-1)(2y+2)$**(0.5 point)**

$$D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2$$

Points	f_{xx}	f_{yy}	$(f_{xy})^2$	$D(x, y)$	Conclusion
$(0, 0)$	0	0	-2	-4	Saddle point **(0.4 point)**
$(1, 0)$	0	0	2	-4	Saddle point **(0.4 point)**
$(0, -2)$	0	0	2	-4	Saddle point **(0.4 point)**
$(1, -2)$	0	0	-2	-4	Saddle point **(0.4 point)**
$(1/2, 1)$	-2	$-1/2$	0	1	Loc. max. **(0.4 point)**