Homework week 6 solutions

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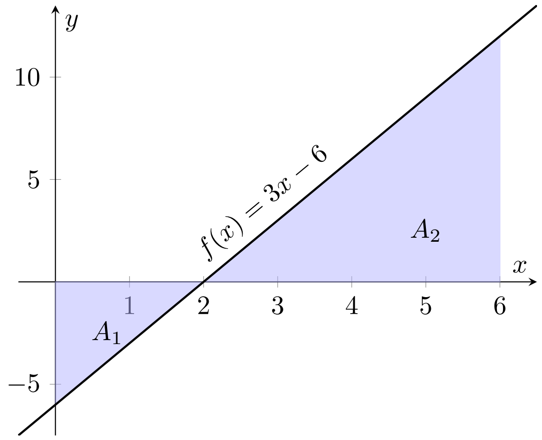
1 Riemann Sum (5 marks)

1. Compute the region between the graph of $y = 3x - 6$ and the x-axis, for $0 \leqslant x \leqslant 6$

   Solution: Here the question asks for the sum of the areas A_1 ans A_2



Warning: If you compute the $\int_0^6 (3x - 6)\ dx$ you get the **net area!!!** between the function f and the $x-$axis (which is equal to $-A_1 + A_2$). So, we have:

$$A_1 = \int_0^2 -f(x)\ dx = \int_0^2 (6 - 3x)\ dx = \left[6x - \frac{3}{2}x^2\right]_0^2 = 6 \quad \textbf{(1 mark)}$$

$$A_2 = \int_2^6 f(x)\ dx = \int_2^6 (3x - 6)\ dx = \left[\frac{3}{2}x^2 - 6x\right]_2^6 = 24 \quad \textbf{(1 mark)}$$

Answer $= 6 + 24 = 30$ **(0.5 mark)**

2. Compute the Midpoint Riemann sum for the function $f(x) = x^2$ on the interval $[-5, 5]$ using $n = 10$ equal subintervals

   Solution: $a = -5, b = 5, n = 10$. So, $\Delta x = \dfrac{b-a}{n} = 1$ **(0.5 mark)**

$$x_k^* = -5 + \frac{\Delta x}{2} + (k-1)\Delta x = -5 + \frac{1}{2} + k - 1 = -5.5 + k \quad \textbf{(0.5 mark)}$$

$$\begin{aligned} M_{10} &= f(-4.5)\Delta x + f(-3.5)\Delta x + f(-2.5)\Delta x + f(-1.5)\Delta x + f(-0.5)\Delta x \\ &+ f(0.5)\Delta x + f(1.5)\Delta x + f(2.5)\Delta x + f(3.5)\Delta x + f(4.5)\Delta x \quad \textbf{(1 mark)} \\ &= 20.25 + 12.25 + 6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25 + 12.25 + 20.25 = 82.5 \end{aligned}$$

$$M_{10} = 82.5 \quad \textbf{(0.5 mark)}$$

2 Substitution or Integration By Parts (1.5 + 1.5 + 2) + 2 bonus

1. Find the derivative $\frac{dF}{dx}$ of the following function:

$$F(x) = \int_{\arctan(x)}^{\ln(x)} (t^2 + 2)dt$$

Do not simplify the answer. **Solution:** We have that:

$$\begin{aligned} F(x) &= \int_{\arctan(x)}^{0} (t^2+1)\ dt + \int_{0}^{\ln(x)} (t^2+1)\ dt \quad \textbf{(0.5 mark)} \\ &= -\int_{0}^{\arctan(x)} (t^2+1)\ dt + \int_{0}^{\ln(x)} (t^2+1)\ dt \end{aligned}$$

Using the Fundamental Theorem of Calculus Part I and chain rule, we get:

$$\frac{dF}{dx} = -(\arctan^2(x)+1)\frac{1}{x+1} + (\ln^2 x + 1)\frac{1}{x} \quad \textbf{(1 mark)}$$

2. Evaluate the following definite integrals:

(a) $\int \theta \sec^2 \theta d\theta$

Solution: Here we use Integration by Parts and apply the ILATE rule

$$\int \underbrace{\theta}_{\text{A}} \underbrace{\sec^2\theta}_{\text{T}} d\theta$$

$$u = \theta \qquad dv = \sec^2 \theta d\theta$$

$$du = d\theta \qquad v = \tan\theta$$

So,

$$\int \theta \sec^2 \theta d\theta = \theta \tan\theta - \int \tan\theta\ d\theta \quad \textbf{(0.5 mark)}$$

Now, we need to evaluate $\int \tan\theta\ d\theta$

$$\int \tan\theta\ d\theta = \int \frac{\sin\theta}{\cos\theta}\ d\theta$$

Using substitution $u = \cos\theta$ and $du = -\sin\theta\ d\theta$, we have:

$$\int \tan\theta\ d\theta = \int -\frac{du}{u}\ du = -\ln|u| + c = -\ln|\cos\theta| + c \quad \textbf{(0.5 mark)}$$

Finally, we have:

$$\int \theta \sec^2\theta d\theta = \theta\tan\theta - \ln|\cos\theta| + c \quad \textbf{(0.5 mark)}$$

(b) $\int \frac{1}{x^2\sqrt{4-x^2}}dx$. Use trigonometric substitution with $x = 2\sin\theta$ and the Pythagorean theorem. Use $\frac{1}{\sin\theta} = \csc\theta$ and all the tables on the lecture notes website. **BONUS (2 marks)**

Solution: Using trigonometric substitution with $x = 2\sin\theta$ and $dx = 2\cos\theta\ d\theta$ we get:

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2 x} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

The integral becomes:

$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{4-x^2}}dx &= \int \frac{1}{8\sin^2\theta\cos\theta}(2\cos\theta\ d\theta) \\
&= \frac{1}{4}\int \frac{1}{\sin^2\theta}\ d\theta = \frac{1}{4}\int \csc^2\theta\ d\theta \\
&= -\frac{1}{\cot\theta} + c \qquad \textbf{(1 mark)}
\end{aligned}
$$

Since $x = 2\sin\theta$, we get $\sin\theta = \frac{x}{2}$. which means that in a right-angled triangle, the opposite side of θ is x and the hypotenuse is 2. Using Pythagorean theorem, we get the adjacent side to θ is $\sqrt{4-x^2}$. So, $\cot\theta$ which is the ratio of adjacent side over the opposite side, equals to $\frac{\sqrt{4-x^2}}{x}$. Hence,

$$
\int \frac{1}{x^2\sqrt{4-x^2}}dx = -\frac{\sqrt{4-x^2}}{4x} + c \qquad \textbf{(1 mark)}
$$

(c) $\int \sin(x)\cos(x)\ln[\sin(x)]dx$

Solution: First, use substitution $t = \sin x$ and $dt = \cos x\ dx$. The integral becomes:

$$
\int \sin(x)\cos(x)\ln[\sin(x)]dx = \int t\cos x\ln(t)\ \frac{dt}{\cos x} = \int t\ln t\ dt \qquad \textbf{(0.5 mark)}
$$

Now, we use Integration by Parts and the ILATE rule **(0.5 mark)**:

$$
\begin{aligned}
u &= \ln t \qquad dv = tdt \\
du &= \frac{1}{t} \qquad v = \frac{t^2}{2} \\
\int t\ln t\ dt &= \frac{t^2}{2}\ln t - \int \frac{t^2}{2}\cdot\frac{1}{t}\ dt = \frac{t^2}{2}\ln t - \int \frac{t}{2}\ dt \\
&= \frac{t^2}{2}\ln t - \frac{1}{2}\frac{t^2}{2} + c = \frac{t^2}{2}\ln t - \frac{1}{4}t^2 + c \qquad \textbf{(0.5 mark)}
\end{aligned}
$$

Now, we put back the value of t to get:

$$
\int \sin(x)\cos(x)\ln[\sin(x)]dx = \frac{\sin^2 x}{2}\ln(\sin x) - \frac{1}{4}\sin^2 x + c \qquad \textbf{(0.5 mark)}
$$

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