Opple Inc. is the only manufacturer of the popular oPad. Opple estimates that when the price of the oPad is \$200, then the weekly demand for it is 5000 units. For every \$1 increase in the price, the weekly demand decreases by 50 units. Assume that the fixed costs of production on a weekly basis are \$100 000, and the variable costs of production are \$75 per unit.

(a) Find the linear demand equation for the oPad. Use the notation p for the unit price and q for the weekly demand.

Solution.

A data point is $(q, p) =$ (quantity, price). So, two points are $(5000, 200)$ and $(4950, 201)$. Since the rate of change is constant, the demand curve must be a line with slope $m = \frac{\Delta p}{\Delta q} = \frac{201-200}{4950-5000} = \frac{1}{-50}$. Note that you can just get the slope directly since $\Delta p = 1$ corresponds to a $\Delta q = -50$.

Thus, $p = -\frac{1}{50}q + K$ where K is some constant. Substitute in $(5000, 200)$ to get $200 = -\frac{1}{50} \cdot 5000 + K$, implying $K = 300$.

Therefore, $\boxed{p = -\frac{1}{50}q + 300}$.

Sometimes it helps to massage this equation around to make p the independent variable:

$$q = -50p + 15000,$$

but this is not necessary here. □

(b) Find the weekly cost function, $C = C(q)$, for producing q oPads per week. Note that $C(q)$ is a linear function.

Solution.
We can pretty safely assume the cost is linear given the information at hand.

So the fixed costs were \$100,000 and the variable costs were \$75 per unit. Hence

$$\boxed{C(q) = 100,000 + 75q}$$

is not too bad! This is a nice simple example, it might not always be so clean. For example — there may be "economies of scale" where the cost per unit drops as we make more units. Or perhaps if we make lots of units then we will need to open a second factory and so our costs will jump. etc etc.

□

(c) Find the weekly revenue function, $R = R(q)$. Note that $R(q)$ is a quadratic function.

Solution.

We know that $R = p \cdot q$, so

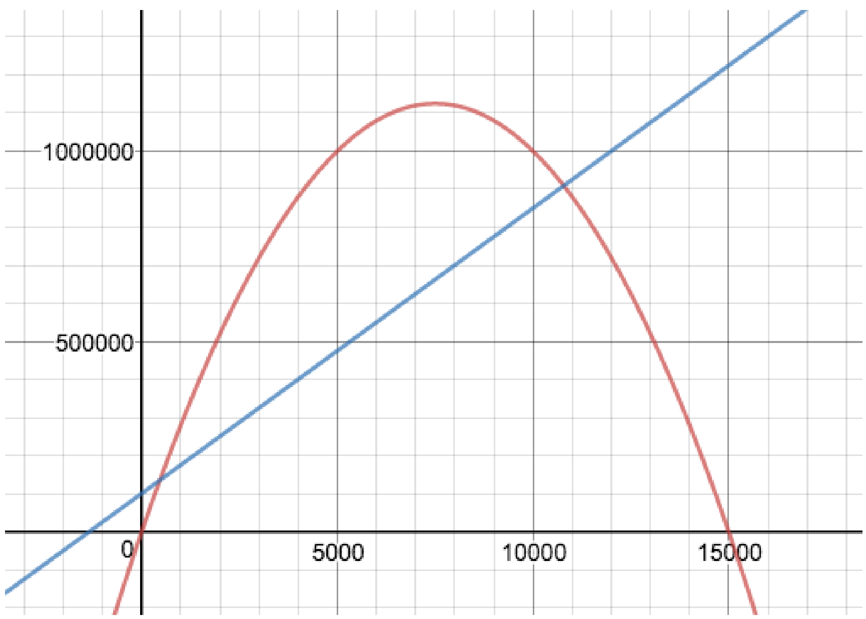
$$\boxed{R(q) = p \cdot q = q\left(300 - \frac{q}{50}\right)}.$$

□

(d) The *break-even* points are where Cost equals Revenue; that is, where $C(q) = R(q)$. Find the break-even points for the oPad.

Solution.

Lets do a simple sketch of revenue and costs



Note that there are 2 intersections — so two break-even points. One where demand is low (and price high) and one where demand is high (and so price is low).

The solution requires some algebra and the quadratic equation:

$$\begin{aligned}
C(q) &= R(q) \\
100000 + 75q &= q\left(300 - \frac{q}{50}\right) \\
100000 + q(75 - 300) + \frac{q^2}{50} &= 0 \\
100000 - 225q + \frac{q^2}{50} &= 0 \\
q &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \boxed{\frac{225 \pm \sqrt{225^2 - 8000}}{2/50}} \\
&= 5625 \pm 125\sqrt{1705} \approx 463.544295, 10786.45570
\end{aligned}$$

Okay this is a bit ugly — but the boxed answer would get you full marks on the exams; this is "calculator ready". This is also what you could enter into WeBWorK. □

(e) On the same set of axes, sketch graphs of $C = C(q)$ and $R = R(q)$ and use these graphs to help you explain why there are two break-even points.

Solution.
We put the graph into part (d). □

(f) *Profit* is defined as Revenue minus Cost: $P(q) = R(q) - C(q)$. Find the profit function $P(q)$. Note that it is a quadratic function.

Solution.

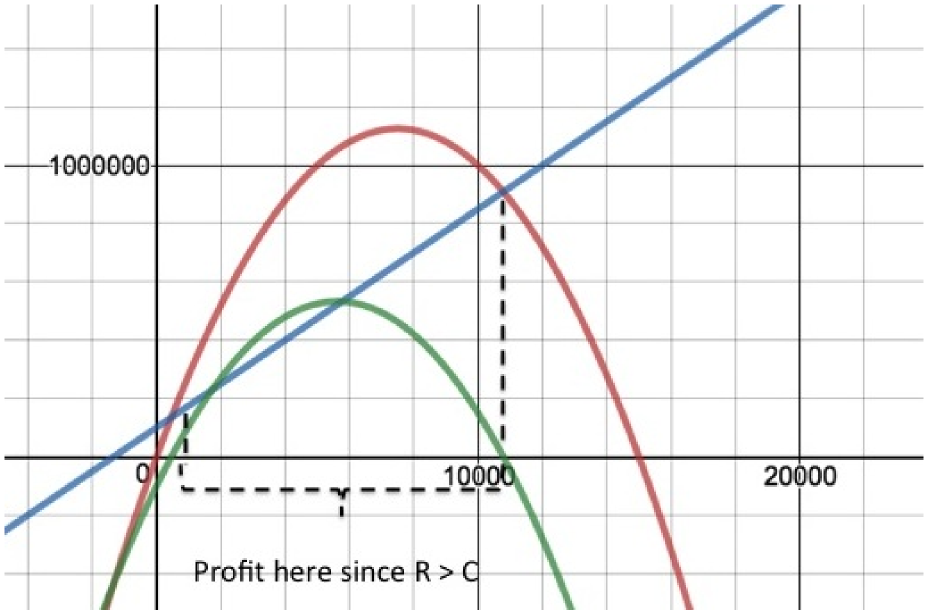
Profit is just Revenue minus Costs (its going to be a quadratic function of q)

$$\begin{aligned}
P(q) &= R(q) - C(q) \\
&= q\left(300 - \frac{q}{50}\right) - (100000 + 75q) \\
&= 300q - \frac{q^2}{50} - 100000 - 75q \\
&= -\frac{q^2}{50} + 225q - 100000
\end{aligned}$$

□

(g) Graph $P = P(q)$ on the same axes as you sketched the graphs of $C(q)$ and $R(q)$. On this graph, indicate the regions of profit ($P(q) > 0$) and loss ($P(q) < 0$).

Solution.



□

(h) How should Opple Inc. operate in order to maximize the weekly profit $P = P(q)$? Use mathematics in your explanation.

Solution.
In order to maximize profit, we'll need to find the vertex of the parabola:

$$p = -\frac{q^2}{50} + 225q - 100000.$$

We can do this one of three ways.

I. **Use Symmetry**:
We know that a parabola is symmetric, making the vertex halfway between the two roots, which we found in part (d).
It looks a little scary, but it's not that bad:

$$\begin{aligned} q_{max} &= \frac{1}{2}\left(q_{BE_1} + q_{BE_2}\right) \\ &= \frac{1}{2}\left[\left(\frac{225 - \sqrt{225^2 - 8000}}{2/50}\right) + \left(\frac{225 + \sqrt{225^2 - 8000}}{2/50}\right)\right] \\ &= \frac{25}{2}\left[\left(225 - \sqrt{225^2 - 8000}\right) + \left(225 + \sqrt{225^2 - 8000}\right)\right] \\ &= \frac{25}{2} \cdot 450 \\ &= 25 \cdot 225 = 5625 \end{aligned}$$

II. **Complete the Square**:
This is painful, but it works..

$$-\frac{q^2}{50} + 225q - 100000 = -\frac{1}{50}(q - 5625)^2 + \frac{1,065,625}{2}$$

If you need a refresher on how to complete the square, visit: https://www.mathsisfun.com/algebr square.html. Note that you can get through this course without having to do this, but sutdents proceeding to Calc II should refresh this technique.

III. **Use Calculus:**

We know that the one, and only, maximum must occur when $\frac{dp}{dq} = 0$. So we need the derivative:

$$\frac{dp}{dq} = \frac{d}{dq}\left(-\frac{q^2}{50} + 225q - 100000\right) = -\frac{q}{25} + 225.$$

Setting this equal to 0 yeilds, $\boxed{q = 25 \cdot 225 = 5625}$.

□

And final word on how to compute the price $p(5625)$. We have to substitute $q = 5625$ into our demand equation to get

$$p = -\frac{5625}{50} + 300.$$

Notice that dividing by 50 is the same as dividing the double by 100. That is,

$$\frac{1}{50} = \frac{2}{100},$$

so you can first double 5625 to get 11250 and then shift the decimal place twice to get 112.50. Therefore,

$$p = 300 - 112.50 = \boxed{187.50}.$$