Exponential + Logarithms

Let's start with a number $a > 0$ (called base)

$$a^0 = 1 \quad , \quad a^1 = a \quad , \quad a^2 = a \cdot a$$

The general rule is $\boxed{a^{x_1 + x_2} = a^{x_1} \cdot a^{x_2}}$

An additional property is $\boxed{\left(a^{x_1}\right)^{x_2} = a^{x_1 x_2}}$

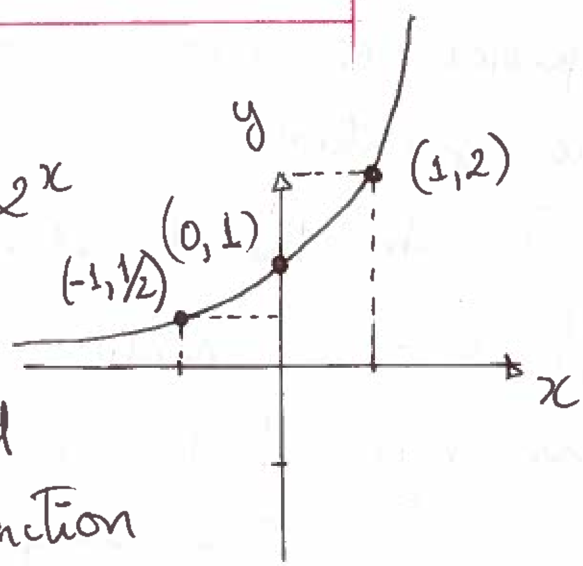
<u>Definition</u>

~~Let a an~~ Let m a real number and $n > 1$ an integer.

$$\boxed{a^{m/n} = \sqrt[n]{a^m}}$$

<u>Example 1</u>

$$f(x) = y = 2^x$$



Here, $f(x)$ is called an exponential function with base 2.

<u>Domain and range of exponential functions</u>

The <u>domain</u> of $f(x) = a^x$ is the entire real line $(-\infty ; +\infty)$

Since $a>0$, a^x will be positive if x is positive. And if $x<0$, then $\boxed{a^x = \frac{1}{a^{-x}}}$ which is the reciprocal of a positive number and ~~also~~ so is still positive. Therefore $y=a^x$ is always positive.

So the range of $f(x)=a^x$ is the set of positive reals. $(0;+\infty)$

Stretching, shifting, Translating exponential functions

Using the previous properties, we can change the form of our function.

Example 2: Describe the transformation used to move the graph of $y=2^x$ onto the graph of $y=2^{x+2}$

Solution: We know that $a^{x_1+x_2} = a^{x_1} \cdot a^{x_2}$, so $y=2^{x+2} = 2^x \cdot 2^2$. The graph $y=2^x$ is then shiftted of stretched by a factor of 2^2 units in the y-direction.

Properties of the exponent

Suppose that $8^a = 3$ and $8^b = 5$. Find the exponent on 8 that gives:

1) 2 2) 15 3) 25 4) 10

Here the unknowns are a and b.

Solutions: Since $8^a = 3$ and $8^b = 5$, then

1) $2 = 8^{1/3}$. So our answer is $\boxed{1/3}$

2) $15 = (3)(5) = (8^a)(8^b) = 8^{a+b}$. So our answer is $\boxed{a+b}$

3) $25 = (5)^2 = (8^b)^2 = 8^{2b}$. So our answer is $\boxed{2b}$

4) $10 = (2)(5) = (8^{1/3})(8^b) = 8^{1/3+b}$. So our answer is $\boxed{\frac{1}{3}+b}$

Solving exponential equations

Now that we have seen the definition of exponential functions we have to solve equations involving them. This can be done using 2 types of method (a simple one and a more complicated one).

The first one will use fact about exponential

$$\boxed{\text{If } b^x = b^y \text{ then } x = y}$$

Note that both expressions have <u>the same base</u> b.

<u>Example 3:</u> Solve the following.

a) $5^{3x} = 5^{7x-2}$

b) $3^t = 9^{t+5}$

c) $4^{5-9x} = \dfrac{1}{8^{x-2}}$

<u>Solution:</u>

a) $3x = 7x - 2$.

$2 = 4x \Rightarrow \boxed{x = \frac{1}{2}}$

b) $3^t = (3^2)^{t+5}$

$3^t = 3^{2(t+5)}$

$\Rightarrow t = 2(t+5) = 2t + 10$

$\boxed{t = -10}$

c) $4^{5-9x} = \dfrac{1}{8^{x-2}}$

$(2^2)^{5-9x} = \dfrac{1}{8^{x-2}} = \dfrac{1}{(2^3)^{x-2}}$

$2(5-9x) = -3(x-2)$

$10 - 18x = -3x + 6$

$\Rightarrow \boxed{x = \frac{4}{15}}$

One-to-one function

A function f with domain A is called a one-to-one function if every $f(x)$-value in the range B comes from only one x-value in A.

In other words, every element in A is connected to a unique element in B.

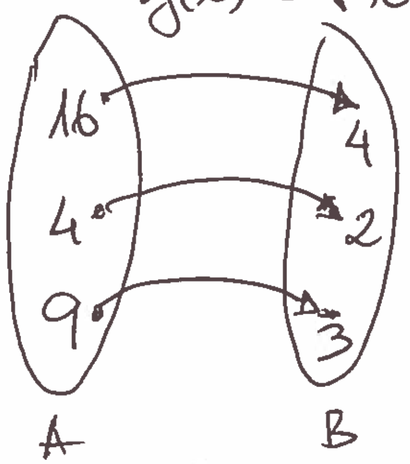
Formally: f is 1-1 if and only if for some $x_1, x_2 \in A$

$$f(x_1) = f(x_2) \text{ implies that } x_1 = x_2 .$$

Example:

One-to-one

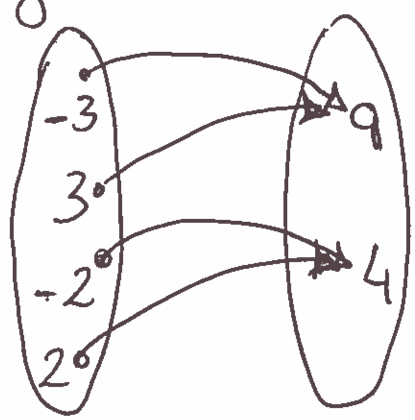
$f(x) = \sqrt{x}$



$A = \{x \in \mathbb{R} \mid x \geq 0\}$

Not one-to-one

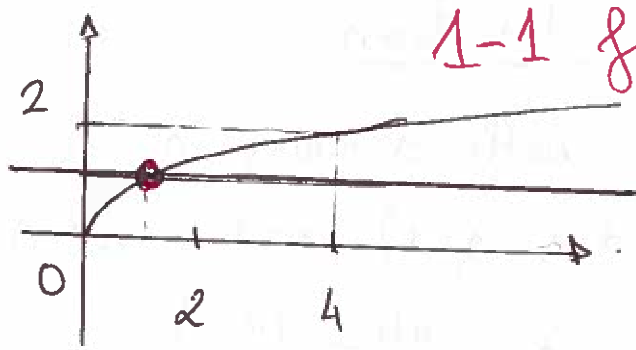
$g(x) = x^2$



$A = \{x \in \mathbb{R}\}.$

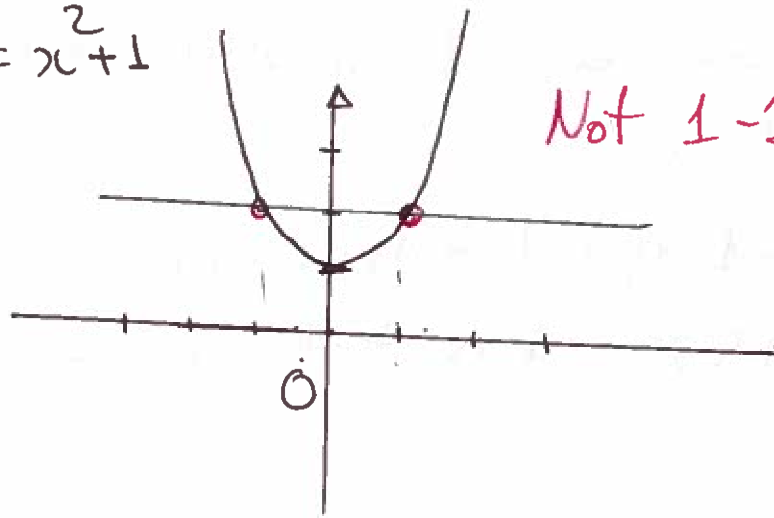
Graphically, we use the Horizontal Line Test to determine such a function. A graph represents a 1-1 function if and only if every horizontal line intersects that graph at most once.

$f(x) = \sqrt{x}$



1-1 function

$g(x) = x^2 + 1$



Not 1-1 function