Exponential + Logarithms (continue)

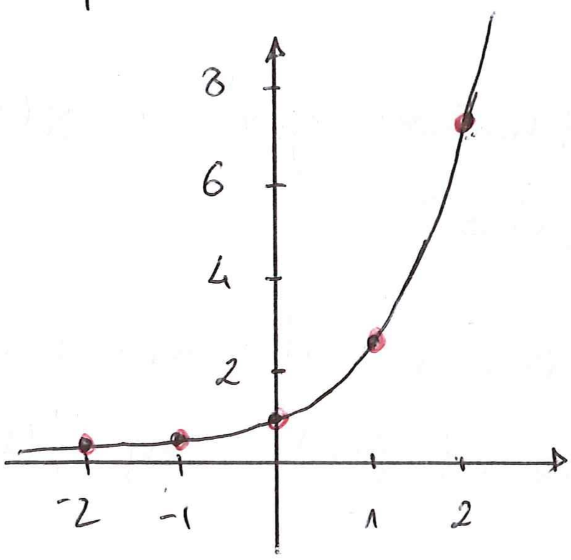
The very specific number e

In this topic, we need to discuss about a special exponential function. In fact it is so special that many people think that this is "THE" exponential function.

$$f(x) = e^x \qquad D = \{x \in \mathbb{R}\}$$

$$e = 2.718281828\ldots \qquad R = \{y \in \mathbb{R} \mid y > 0\}$$

Example 1: Sketch the graph of $f(x) = e^x$



x	-2	-1	0	1	2
$f(x)$	0.13	0.36	1	2.72	7.39

Inverse function

Let f and g 2 functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f.

The function g is called the inverse function of f and is noted by f^{-1}. The domain of f must be equal to the range of f^{-1}. the graph of f^{-1} is the symmetric graph of $f(x)$ with respect to the line $y = x$.

Example 2:

$$f(x) = 3x - 2 \; ; \; g(x) = \frac{x}{3} + \frac{2}{3}$$

$$f(-1) = -5 \quad \Rightarrow \quad g(-5) = -1$$

$$g(2) = \frac{4}{3} \qquad\qquad f\left(\frac{4}{3}\right) = 2$$

Given two one-to-one functions $f(x)$ and $g(x)$ if $(f \circ g)(x) = x$ AND $(g \circ f)(x) = x$ then $f(x)$ and $g(x)$ are inverses of each other. More specifically $g(x)$ is the inverse of $f(x)$. (resp. $f(x)$)

$$g(x) = f^{-1}(x).$$

Notation $(f \circ g)(x) = f(g(x))$

Here $f^{-1}(x) \neq \dfrac{1}{f(x)}$

Find the inverse of a function.

Example 3: Given $f(x) = 3x - 2$, find $f^{-1}(x)$.

1 - First replace $f(x)$ with y:

$$y = 3x - 2$$

2 - Replace all x with y and all y with x.

$$x = 3y - 2$$

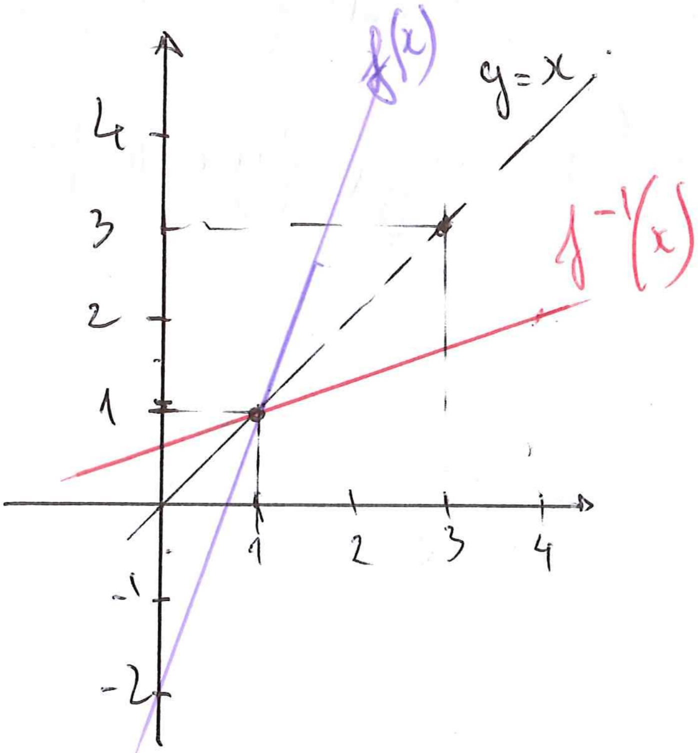
3 - Solve for y: $x + 2 = 3y \Rightarrow y = \frac{x}{3} + \frac{2}{3}$.

$$\boxed{f^{-1}(x) = \frac{x}{3} + \frac{2}{3}}$$

Now, we need to verify the result. It means, we check if $(f \circ f^{-1})(x) = x$.

$$(f \circ f^{-1})(x) = f\left[f^{-1}(x)\right] = f\left[\frac{x}{3} + \frac{2}{3}\right] = 3\left(\frac{x}{3} + \frac{2}{3}\right) - 2$$

$$= x + 2 - 2 = x \qquad \text{TRUE}$$



Example 4: Let f be the function $f(x) = 1+x^2$. Is f invertible?

Let $g(x) = 1+x^2$ with the domain $[0, +\infty[$ and give the graph of the inverse function and the domain and range.

Solution: f not invertible. For example, $f(x)=5$ has 2 solutions, $x=2$ and $x=-2$

Now, if we restrict to $x \in [0;+\infty[$, then $x^2+1=y$ has one solution when $y \geq 1$ and zero solutions if $y<0$

$$x = \sqrt{y-1}$$

Thus the inverse function g^{-1} exists and $g^{-1}(x) = \sqrt{x-1}$

$\mathcal{D}_g = \{x \in \mathbb{R} \mid x \geq 0\}$ | $\mathcal{D}_{g^{-1}} = \{x \in \mathbb{R} \mid x \geq 1\}$

$\mathcal{R}_g = \{y \in \mathbb{R} \mid y \geq 1\}$ | $\mathcal{R}_{g^{-1}} = \{y \in \mathbb{R} \mid y \geq 0\}$.

Logarithmic Functions

The logarithmic function $g(x) = \log_b(x)$ is the inverse function $f(x) = b^x$. And so the meaning of $y = \log_b(x)$ is $b^y = x$. The expression $b^y = x$ is said to be the exponential form for the logarithm $y = \log_b(x)$

example 5

1. $\log_2(8) = x$.

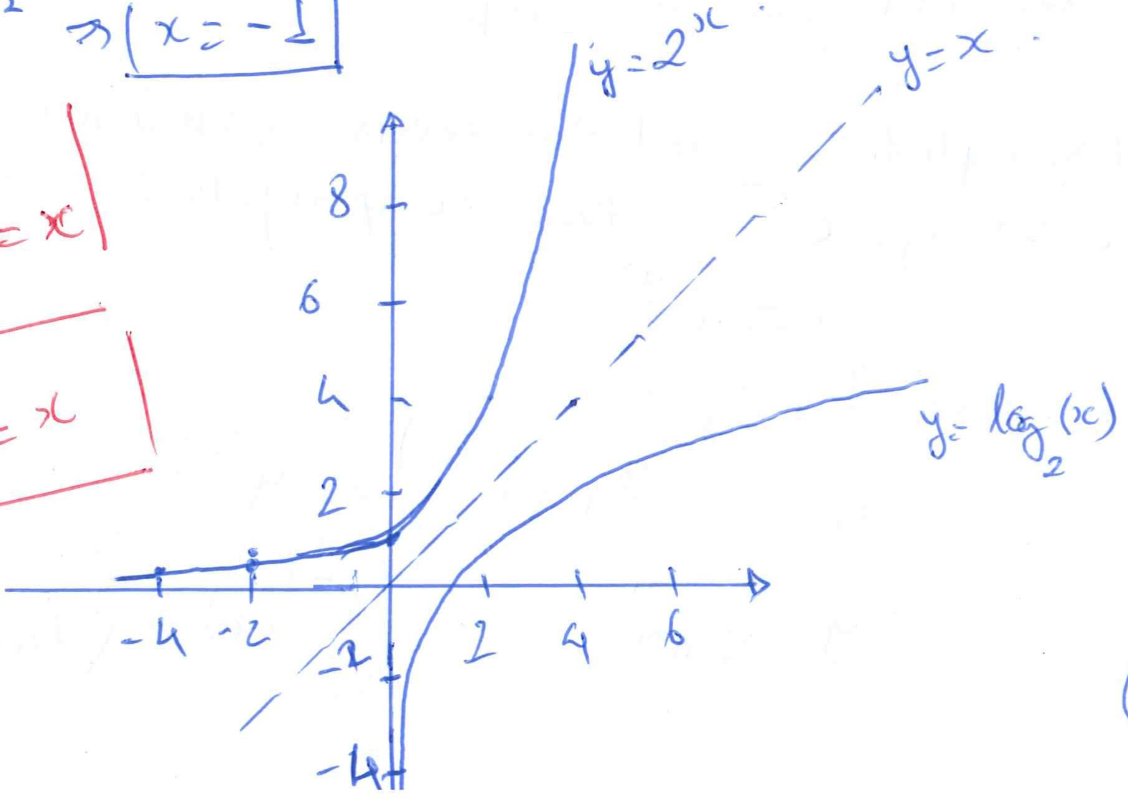
Solution: the exponential form is $2^x = 8$.
Since $8 = 2^3 \Rightarrow 2^x = 2^3 \Rightarrow \boxed{x = 3}$

2. $\log_2\left(\frac{1}{2}\right) = x$

Solution: the exponential form is $2^x = \frac{1}{2}$. Since $\frac{1}{2} = 2^{-1} \Rightarrow \boxed{x = -1}$

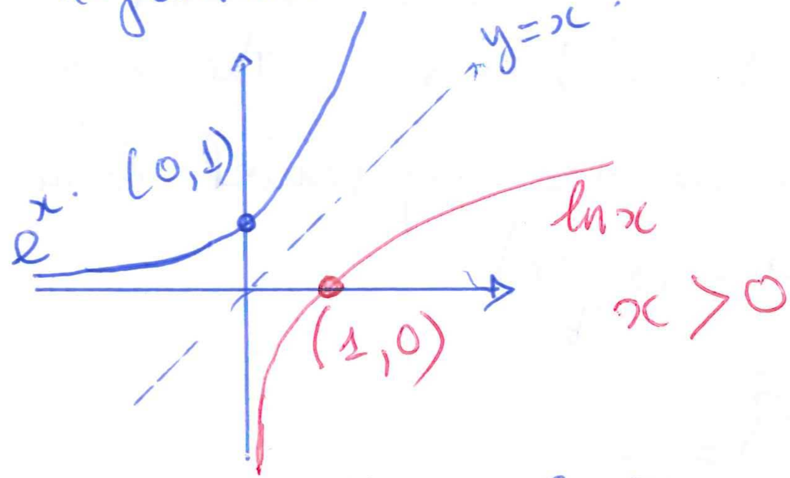
$$\boxed{\log_b(b^x) = x}$$

$$\boxed{b^{\log_b(x)} = x}$$



Using the special base e, we have

$y = e^x \Leftrightarrow \log y = x \Rightarrow \boxed{\ln y = x}$. It defines the natural logarithm.



Properties of logarithmic functions.

- $\log_b (MN) = \log_b (M) + \log_b (N)$.
- $\log_b \left(\frac{M}{N}\right) = \log_b (M) - \log_b (N)$.
- $\log_b (M^c) = c \log_b M$.

Example 6: Find the inverse function of $f(x) = e^{x^2}$

- Set $y = e^{x^2}$, then swap inputs:

$$x = e^{y^2}.$$

- Solve for y:

$$\ln x = \ln (e^{y^2}) = y^2.$$

$$\Rightarrow y = \sqrt{\ln x} \Rightarrow \boxed{f^{-1}(x) = \sqrt{\ln(x)}}$$