CONTINUITY

Sep. 18

Definition: A function is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

If a function is not continuous at a then it is said to be discontinuous at a.

So if a function is continuous at $x = a$ we immediately know that

- $f(a)$ exists
- $\lim_{x \to a^-}$ exists and is equal to $f(a)$ and
- $\lim_{x \to a^+}$ exists and is equal to $f(a)$.

Definition: A function $f(x)$ is continuous from the right at a if:

$$\lim_{x \to a^+} f(x) = f(a)$$

Similarly, a function $f(x)$ is continuous from the left at a if:

$$\lim_{x \to a^-} f(x) = f(a)$$

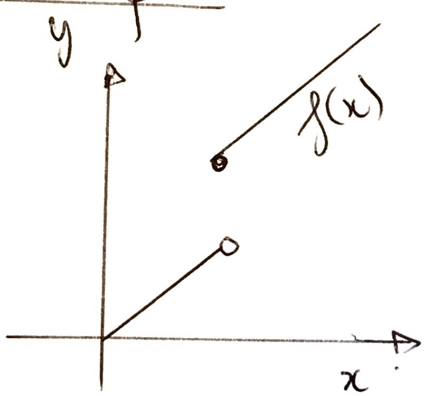
Definition: A function $f(x)$ is continuous on the closed interval $[a,b]$ when:

- $f(x)$ is continuous on (a,b)
- $f(x)$ is continuous from the right at a, and
- $f(x)$ is continuous from the left at b.

The last two conditions are equivalent to:

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b)$$

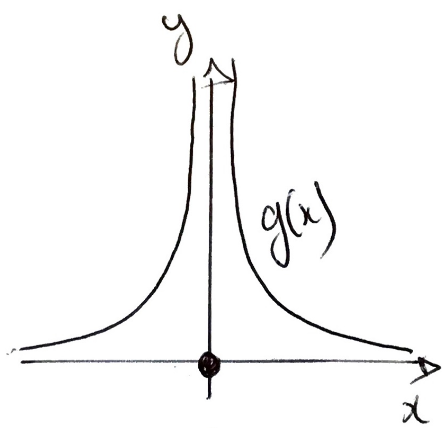
Example 1:



$$f(x) = \begin{cases} x & x < 1 \\ x+2 & x \geq 1 \end{cases}$$

Discontinuous

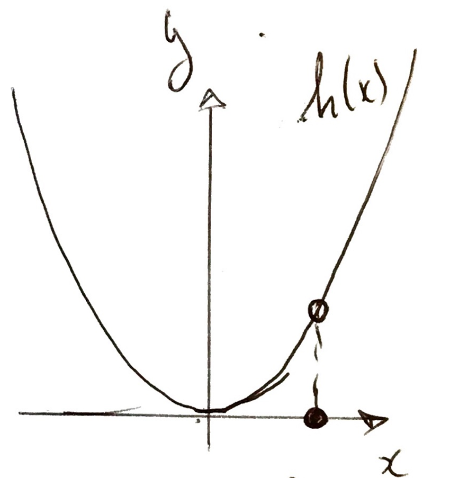
⇒ "jump"



$$g(x) = \begin{cases} 1/x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Discontinuous

⇒ "infinite"



$$h(x) = \begin{cases} \dfrac{x^3 - x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$

Discontinuous.

⇒ "removable"

These functions are continuous ~~in~~ in their domain:

- Polynomials, rationals.
- Exponentials, logarithmic functions
- Trigonometric functions

Continuity on an interval.

(a, b) or $]a, b[$

$[a, b]$

- $f(x)$ is continuous on (a, b) if $f(x)$ is continuous for all x $a < x < b$
- $f(x)$ is continuous on $[a, b]$ if $\cdot$ f is continuous from right at $x = a$, f is continuous from left at $x = b$.

Laws of continuity.

$f(x)$ and $g(x)$ continuous at $x = a$.

- $f \cdot g$ is also continuous at $x = a$.
- $f + g$ " "
- $f - g$ " "
- f/g " " if $g(a) \neq 0$.

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ex: $f(x) = \dfrac{\sin x}{2 + \cos x}$. Is $f(x)$ continuous?

- check if the numerator is continuous, the denominator is continuous and $\neq 0$.

- $\sin(x)$ is continuous on its domain.
- $\cos(x)$. ——— " ———.
- $-1 \leq \cos x \leq 1$
- $1 \leq 2 + \cos(x) \leq 3$. $\Rightarrow$ denominator $\neq 0$.

$\Rightarrow$ $f(x)$ continuous everywhere.

Composition and continuity.

If f is continuous at b and $\lim_{x \to a} g(x) = b$,

then $\lim_{x \to a} f(g(x)) = f(b)$.

It means $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$.

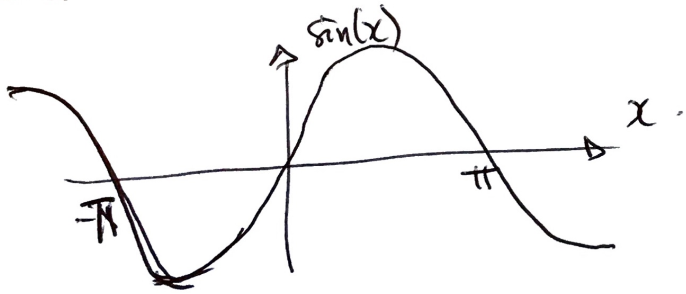
example 2: Where are the following functions continuous?

$f(x) = \sin\left(x^2 + \cos(x)\right)$

$g(x) = \sqrt{\sin(x)}$

- The function $f(x)$ is the composition of $\sin(x)$ with $(x^2 + \cos(x))$
- $\sin x$, x^2, $\cos x$ are continuous on their domains.
- So is the sum $x^2 + \cos x$
- Hence the composition of $\sin(x)$ and $x^2 + \cos(x)$ is continuous everywhere

The second function $g(x)$ is the composition of $\sqrt{x}$ with $\sin x$.

- $\sqrt{x}$ is continuous on its domain $x \geq 0$
- $\sin x$ is continuous, but it is negative in many places.
- 

- Hence $\sin x \geq 0$ $\forall x \in [0, \pi]$ or $x \in [2\pi, 3\pi], \ldots$

To be more precise, $\sin(x) \geq 0$ $\forall x \in [2n\pi, (2n+1)\pi]$ $\forall n$

- Hence $g(x)$ is continuous when $x \in [2\pi n, (2n+1)\pi]$ $\forall n$