# Differentiation

Sep. 25

It is the action of computing a derivative

**Recall:**

Properties of differentiation:

1) $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

or $= \frac{d}{}$

no "prime" ( ' )

$\frac{d}{dx}[f(x) \pm g(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$.

2) $[c\,f(x)]' = c f'(x)$ or $\frac{d}{dx}(c f(x)) = c\frac{df(x)}{dx}$.

**Rules of differentiation.** ⇒ do not use limit definition.

* Power rule

- $f(x) = n x^n, n \neq 0$.
- $f'(x) = n x^{n-1}$.

- $f(x) = [U(x)]^n$
- $f'(x) = n U'(x) U(x)^{n-1}$

Examples:

$f(x) = x^4 \Rightarrow f'(x) = 4x^{4-1} = 4x^3$.

$g(x) = x^{-4} \Rightarrow f'(x) = -4x^{-5}$

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$$h(x) = \left(x^3 + 2x^{-3} + 1\right)^2$$

$$h'(x) = 2\left(x^3 + 2x^{-3} + 1\right)' \left(x^3 + 2x^{-3} + 1\right)^1$$

$$= 2\left(3x^2 - 6x^{-4}\right)\left(x^3 + 2x^{-3} + 1\right)^1$$

$$= 2\left(3x^5 + \cancel{6x^{-1}} + 3x^2 - \cancel{6x^{-1}} - 12x^{-7} - 6x^{-4}\right).$$

$$h'(x) = 2\left(3x^5 + 3x^2 - 6x^{-4} - 12x^{-7}\right)$$

## * Sum rule

- $\left[f(x) + g(x)\right]' = f'(x) + g'(x).$

**example 2:**

$$f(x) = x^6 + x^3 + x^2 + x + 10$$

$$f'(x) = 6x^5 + 3x^2 + 2x + 1$$

## * Difference rule:

$$\left[f(x) - g(x)\right]' = f'(x) - g'(x)$$

**example 3:**

$$f(x) = x^6 - x^3 - x^2 - x - 10$$

$$f'(x) = 6x^5 - 3x^2 - 2x - 1$$

* Constant multiple rule:

$$[c f(x)]' = c f'(x)$$

example 4:

$$f(x) = 2x^{-5}$$

$$f'(x) = -10x^{-6}$$

* I can also combine all these rules.

example 5:

$$f(x) = 3x^{-2} - x^4 + \frac{x^{-5}}{2} + 10.$$

$$f'(x) = (3x^{-2})' - (x^4)' + \left(\frac{x^{-5}}{2}\right)' + \cancel{(10)'} \to 0$$

$$f'(x) = -6x^{-3} - 4x^3 - \frac{5}{2}x^{-6}$$

Derivative of an exponential function

Let $a > 0$ and set $f(x) = a^x$. (exp. func.)

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}.$$

$$= \lim_{h \to 0} a^x \underbrace{\left(\frac{a^h - 1}{h}\right)}_{c(a)}$$

Let's assume that the last limit is equal to $c(a)$

$$c(a) = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

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$\Rightarrow \frac{d}{dx} f(x) = \frac{d a^x}{dx} = C(a) \cdot a^x$

The derivative of $a^x$ is $a^x$ multiplied by some constant. i.e. $a^x$ is nearly unchanged by differentiating.

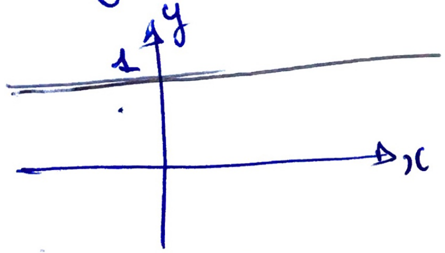
We can notice that we have got the exact definition of the derivative of $f(x) = a^x$ at $x = 0$ $f'(0)$

$\Rightarrow f'(x) = f'(0)\, a^x$

where $f'(0) = \lim_{h \to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}$.

Let's find out the value of $C(a)$.

- If $a = 1$, $\lim_{h \to 0} \frac{1^h - 1}{h} = 0$, since $1^x = 1$




- If $a = 2$ $\Rightarrow \lim_{h \to 0} \frac{2^h - 1}{h}$

| $h$ | 0.1 | 0.01 | 0.001 | 0.0001 | 0.00001 | 0.000001 |
|---|---|---|---|---|---|---|
| $\frac{2^h - 1}{h}$ | 0.717 | 0.6956 | 0.6934 | 0.6932 | 0.6931 | 0.6931 |

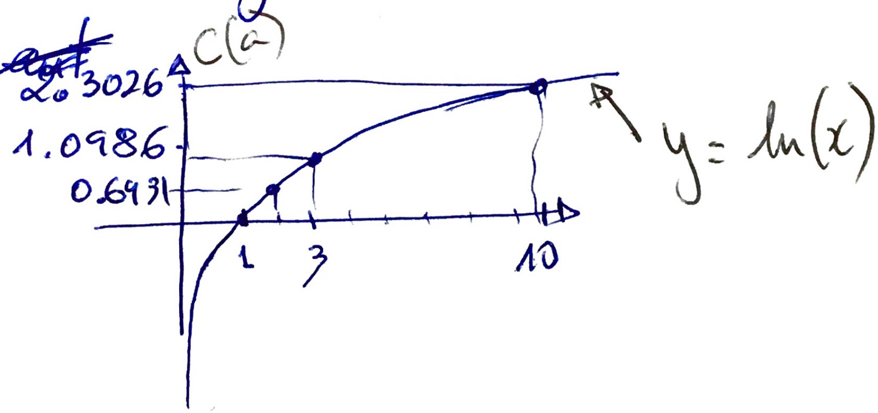
• If ~~c(a)~~ $a = 3$, $\lim_{h \to 0} \frac{3^h - 1}{h}$.

| $h$ | 0.1 | 0.001 | 0.0001 | 0.00001 | 0.000001 |
|---|---|---|---|---|---|
| $\frac{3^h-1}{h}$ | 1.1612 | 1.1047 | 1.0992 | 1.0986 | 1.0986 |

• If $a = 10$

| $h$ | 0.1 | 0.01 | 0.001 | 0.0001 | 0.00001 | 0.000001 |
|---|---|---|---|---|---|---|
| $\frac{10^h-1}{h}$ | 2.5893 | 2.3293 | 2.3052 | 2.3028 | 2.3026 | 2.3026 |

From these values of $a$, $c(a)$ increases as we increase $a$.




So $c(a) = \ln(a)$

Therefore $\frac{df(x)}{dx} = \frac{da^x}{dx} = a^x \ln(a)$

Instead of "a", let's use the exponential function "e"

$$f(x) = e^x \Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h}\right).$$

the constant e is the unique real number that satisfies.

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

But where does that "e" come from?

- We said that

$$c(a) = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

- If we assume that $c(a) = 1 \left( \Rightarrow \frac{d a^x}{dx} = a^x \right)$.

- in the previous example $C(10) \simeq 2.3026$.

$$\Rightarrow a = 10^{\log_{10} a}. \text{ and so } a^h = 10^{h \log_{10} a}$$

$$c(a) = \lim_{h \to 0} \frac{1}{h} \left( 10^{h \log_{10} a} - 1 \right) \qquad \Rightarrow H = h \log_{10} a.$$

$$c(a) = \lim_{H \to 0} \frac{\log_{10} a}{H} \left( 10^H - 1 \right) = \log_{10} a \left( \lim_{H \to 0} \frac{10^H - 1}{H} \right) \quad C(10)$$

$c(a) = \log_{10} a \; c(10)$

If $c(a) = 1 \Rightarrow 1 \simeq \log_{10} a \; c(10)$.

$2.3026 \log_{10} a \simeq 1$

$\log_{10} a \simeq \dfrac{1}{2.3026} \simeq 0.4343$.

$a \simeq 10^{0.4343} \simeq 2.7813$.

$\Rightarrow a \simeq 2.718281828459 05\ldots = e$

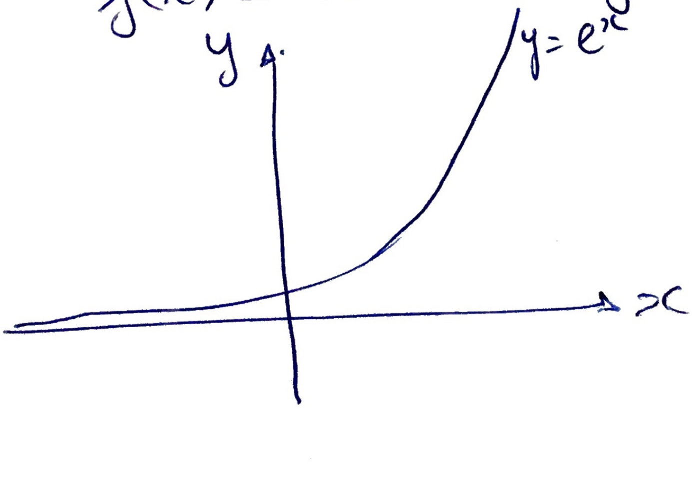
We say that "e" is the unique real number that satisfies $c(a) = 1$ i.e. $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$

So, provided we are using the natural exponential function, we get:

$$f(x) = e^x \Rightarrow f'(x) = e^x$$

But for a general exponential function, we have:

$$f(x) = a^x \Longrightarrow f'(x) = a^x \ln(a).$$




$e^0 = 1$

$\lim_{x \to +\infty} e^x = \infty$

$\lim_{x \to -\infty} e^x = 0$.

## * Product rule.

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'\cdot g + f\cdot g'$$

example 6:

- $f(x) = x \sin x$ $\qquad u = x \quad u' = 1$

$\qquad v = \sin x \quad v' = \cos x.$

$$f'(x) = 1\cdot \sin x + x \cos x = \sin x + x\cos x.$$

- $f(x) = x^2 \cos x$ $\qquad u = x^2 \quad u' = 2x$

$\qquad v = \cos x \quad v' = -\sin x$

$$f'(x) = 2x\cos x - x^2 \sin x$$

## * Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'g - fg'}{g^2}$$

example 7:

- $h(x) = \left(\frac{x^3}{x^6}\right)$ where $f(x) = x^3$ $g(x) = x^6$

$$h'(x) = \left(\frac{x^3}{x^6}\right)' = \left(\frac{1}{x^3}\right)' = \frac{0\cdot x^3 - 1\cdot(3x^2)}{x^6} = -\frac{3x^2}{x^6} = -\frac{3}{x^4}$$