Maximum & Minimum Values

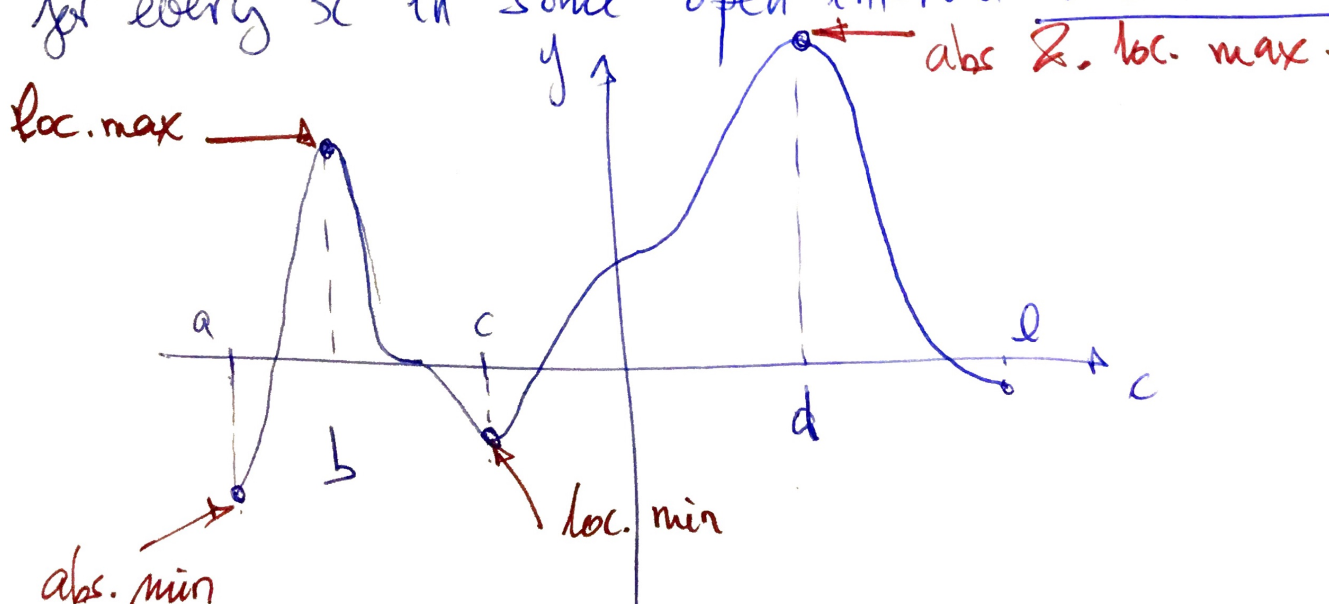
Definition

1. $f(x)$ has an absolute max at $x = c$ if $f(x) \le f(c)$ for every x in the domain of f.

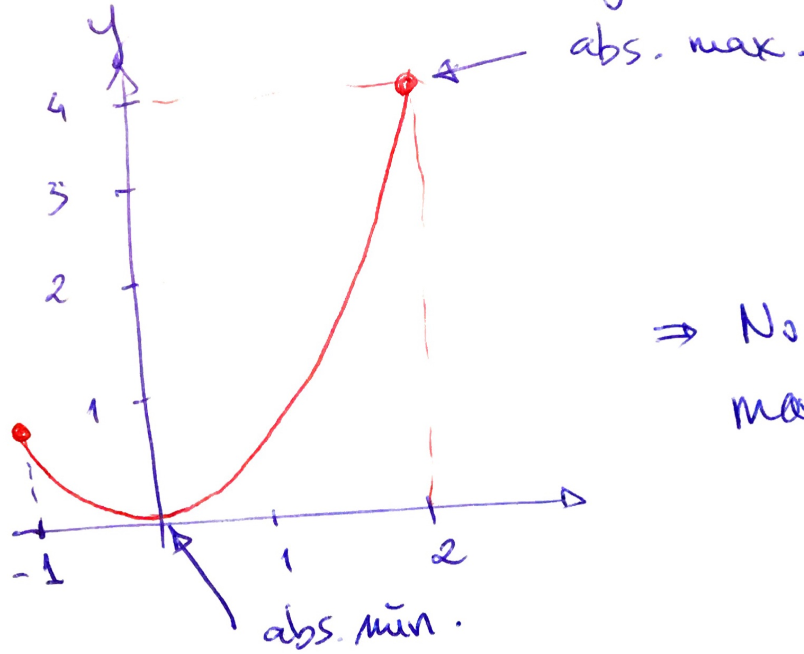
2. $f(x)$ has a local maximum at $x = c$ if $f(x) \le f(c)$ for every x in some open interval around c

3. $f(x)$ has an absolute min at $x = c$ if $f(x) \ge f(c)$ for every x in the domain of f.

4. $f(x)$ has a local min at $x = c$ if $f(x) \ge f(c)$ for every x in some open interval around $x = c$



example 1: Find the absolute extrema and relative extrema for $f(x) = x^2$ on $[-1, 2]$



⇒ No relative maximums.

example 2: Find the minimum and maximum local at $x = -\frac{3}{5}$ and $x = \frac{1}{3}$ for

$$f(x) = 5x^3 + 2x^2 - 3x$$

- First derivative (slope):

$$\frac{dy}{dx} = 15x^2 + 4x - 3$$

It is quadratic with zeros at $x = -3/5$ and $x = 1/3$

Maximum or minimum?

- Second Derivative test:

$$y'' = 30x + 4$$

* at $x = -3/5$

$y'' = 30\left(-3/5\right) + 4 = -14 < 0 \Rightarrow$ loc. max

* at $x = 1/3$

$y'' = 30\left(1/3\right) + 4 = 14 > 0 \Rightarrow$ loc. min.

Second Derivative Test

When the slope of $f(x)$ is zero at x, and the second derivative test at x is:

- $f''(x_0) > 0 \Rightarrow$ loc. ~~max~~ min.
- $f''(x_0) < 0 \Rightarrow$ loc. max.
- $f''(x_0) = 0 \Rightarrow$ inconclusive.

Critical Points

We say that $x = c$ is a critical point of the function $f(x)$ if $f(c)$ exists and if either of the following are true

$f'(c) = 0$ OR $f'(c)$ does not exist.

Note that we require that $f(c)$ exists in order for $x = c$ to actually be a critical point.

example 3: Determine all critical points for

$f(x) = 6x^5 + 33x^4 - 30x^3 + 100$

Solution:

- First derivative.

$$f'(x) = 30x^4 + 132x^3 - 90x^2$$
$$= 6x^2(5x^2 + 22x - 15)$$
$$= 6x^2(5x-3)(x+5)$$

- The only critical points are the values of x that make $f'(x) = 0$.

$$6x^2(5x-3)(x+5) = 0.$$

$$x = 0 \quad , \quad x = \frac{3}{5} \quad , \quad x = -5$$

example 4: Determine all critical points for $y = 6x - 4\cos(3x)$.

Solution:

- First derivative:

$$y' = 6 + 12\sin(3x).$$

- $f'(x) = 0$.

$$6 + 12\sin(3x) = 0.$$
$$\sin(3x) = -\tfrac{1}{2}$$
$$3x = \arcsin\left(-\tfrac{1}{2}\right) + 2\pi n$$

$\Rightarrow \quad 3x = -\frac{5\pi}{6} + 2\pi n \quad$ or $\quad 3x = -\frac{\pi}{6} + 2\pi n$

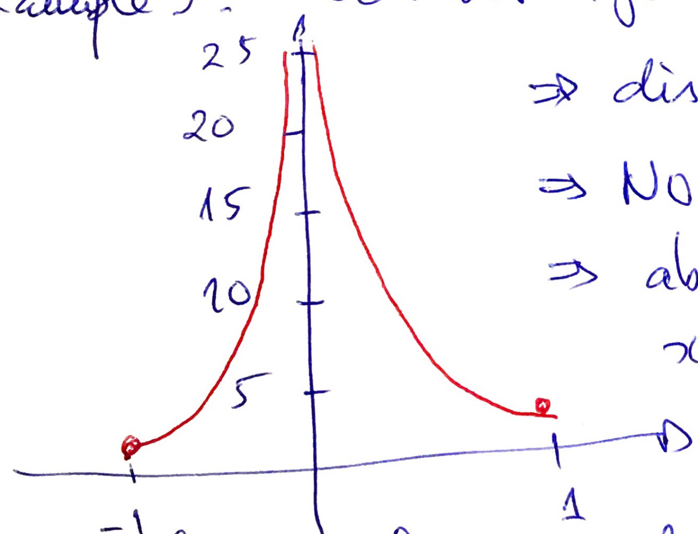
Extreme Value Theorem

Suppose that $f(x)$ is continuous on the in $[a, b]$, then there are two number $a \leq c$, $d \leq b$ so that $f(c)$ is an absolute max. for the function and $f(d)$ in an absolute min. for the function.

So if we have a continuous function on an interval $[a, b]$, then we have abs. max. and abs. min. somewhere in the interval.

!!! * this theorem requires a continuous function.

example 5: Consider $f(x) = \frac{1}{x^2}$ on $[-1, 1]$.



$\Rightarrow$ discontinuous at $x = 0$,

$\Rightarrow$ No abs. max.

$\Rightarrow$ abs. min. at $x = -1$ and $x = 1$

If we change the interval to $[\frac{1}{2}, 1]$, the function has both ~~ex~~ absolute extrema. ~~the may only run into problems~~

Problems $\equiv$ discontinuity.

Fermat theory

If $f(x)$ has a local minimum or local maximum at $f(c)$, and if $f'(c)$ exists, then $f'(c)=0$

i.e. If you want to find local extrema, then look at the points where

$$\begin{cases} f'(x)=0 \\ f'(x) \text{ not defined.} \end{cases}$$