Approximation of functions

- 0^{th} approximation. (constant approximation)

→ The simplest functions are those that are constants.

$$F(x) = A.$$

To ensure that $F(x)$ is a good approximation for x close to "a", we choose A so that $f(x)$ and $F(x)$ take exactly the same value when $x = a$.

$$F(x) = A \Rightarrow F(a) = A = f(a) \Rightarrow A = f(a).$$

- ~~First approximati~~

example: Use the constant approximation to estimate $e^{0.1}$

- First set $f(x) = e^x$.
- Pick a point $x = a$ to approximate the function.
- $F(x) = f(0) = e^0 = 1$

$F(0.1) = 1$ Note that $e^{0.1} = 1.1051709\ldots$

- **First approximation (linear approximation)**

This improves the 0^{th} approximation. We allow $F(x)$ to be the form of $A + Bx$, for some A and B

A and B are constants.

$$F(x) = A + Bx \Rightarrow F(a) = A + Ba = f(a).$$

$$F'(a) = B = f'(a)$$

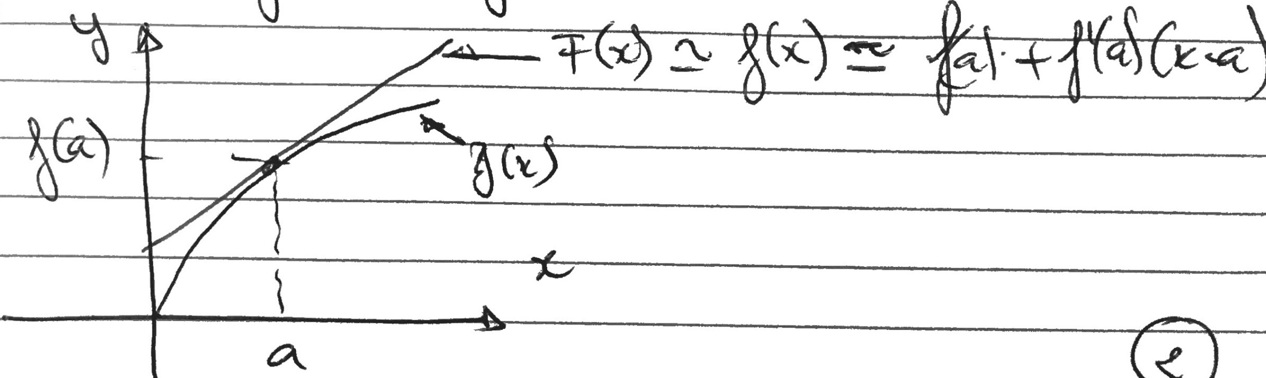
So we must have $B = f'(a)$. Substituting into $A + Ba = f(a)$, we get $A = f(a) - a f'(a)$.

$$F(x) = A + Bx = f(a) - a f'(a) + f'(a) \cdot x.$$

$$F(x) = f(a) + f'(a)(x-a)$$

This first approximation is called linear approximation of $f(x)$ about $x = a$.

$$f(x) \simeq f(a) + f'(a)(x-a).$$



example Use the linear approximation to estimate $e^{0.1}$

- First, set $f(x) = e^x$ and $a = 0$.
- compute $f(a)$ and $f'(a)$

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1.$

$F(x) = f(0) + x f'(0) = 1 + x.$

$F(0.1) = 1.1. \qquad e^{0.1} = 1.1051709 \ldots$

- Second approximation (quadratic approximation).

Use of quadratic function of x.

$F(x) = A + Bx + Cx^2$ $A, B,$ and C are constants.

- $f(a) = F(a)$
- $f'(a) = F'(a)$ and $f''(a) = F''(a)$.
- $f''(a) = F''(a)$

$F(x) = A + Bx + Cx^2 \Rightarrow F(a) = A + Ba + Ca^2 = f(a)$

$F'(x) = B + 2Cx \rightarrow F'(a) = B + 2Ca = f'(a)$

$F''(x) = 2C. \Rightarrow F''(a) = 2C = f''(a)$

Solve for C, B and A.

$$C = \frac{1}{2} f''(a)$$

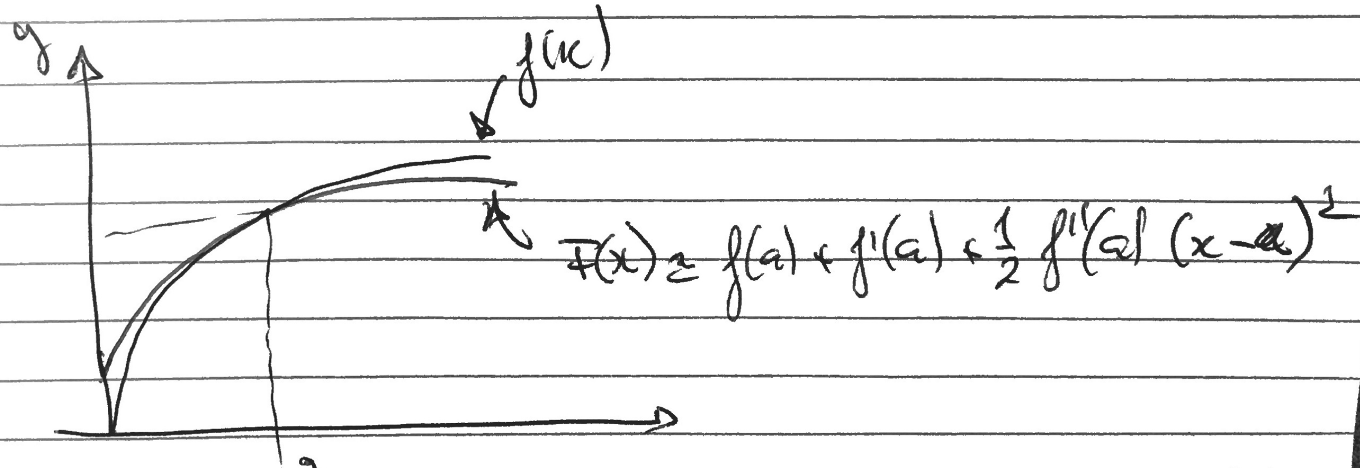
$$B = f'(a) - 2Ca = f'(a) - a f''(a)$$

$$A = f(a) - Ba - Ca^2 = f(a) - a\left[f'(a) - af''(a)\right] - \frac{1}{2} f''(a) a^2$$

Put back together to build up $F(x)$

$$F(x) = f(a) - f'(a)a + \frac{1}{2} f''(a) a^2 + f'(a)x - f''(a)ax$$
$$+ \frac{1}{2} f''(a) x^2$$
$$= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

$$\boxed{f(x) \simeq f(a) + f'(a) + \frac{1}{2} f''(a)(x-a)^2.}$$



example: Use quadratic approximation to estimate $e^{0.1}$

Set $f(x) = e^x$ and $a = 0$.

$f(x) = e^x$ $\quad\quad\quad\quad$ $f(0) = 1$

$f'(x) = e^x$ $\quad\quad\quad\quad$ $f'(0) = 1$

$f''(x) = e^x$ $\quad\quad\quad\quad$ $f''(0) = 1$.

Then we have :

$$F(x) = f(0) + x f'(0) + \frac{1}{2} x^2 f''(0) = 1 + x + \frac{x^2}{2} .$$

$F(0.1) = 1.105$ $\quad\quad$ $e^{0.1} = 1.105170918 \ldots$