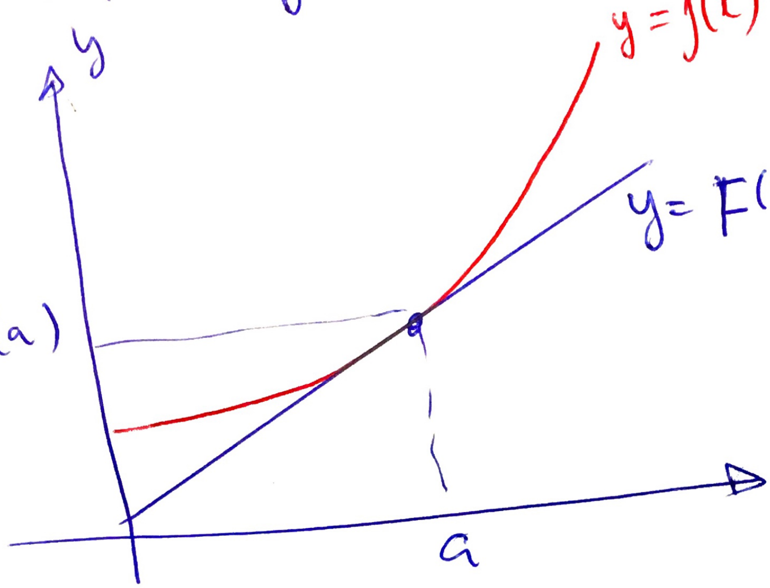
Linear approximations

Given a function $f(x)$, we can find its tangent at $x=a$.

$$F(x) = f(a) + f'(a)\,(x-a).$$



We can see that near $x=a$, the tangent line and the function have nearly the same graph.

Example: Determine the linear approximation for $f(x) = \sqrt[3]{x}$ at $x=8$. Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$

Solution:

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}} \qquad f(8) = 2$$

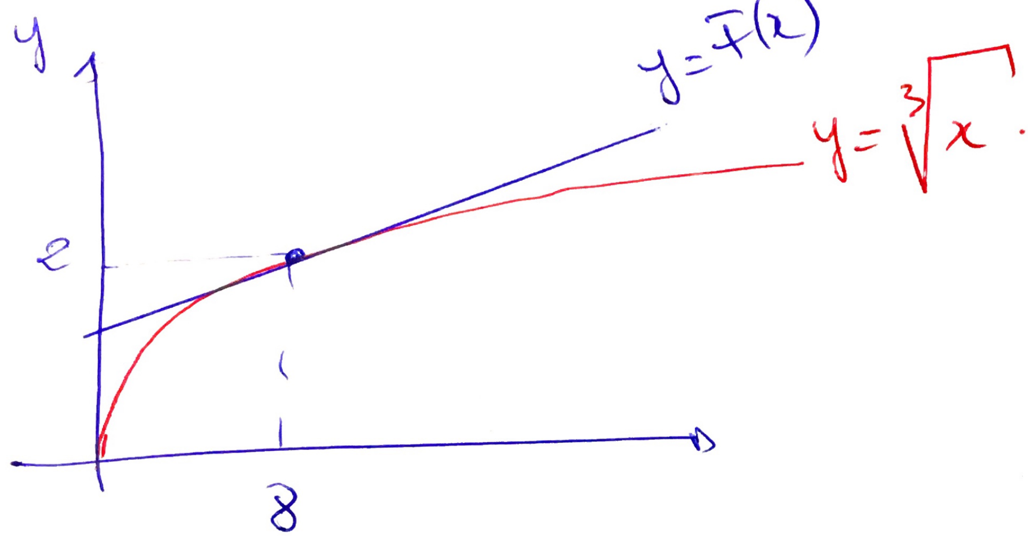
$$f'(8) = \frac{1}{12}$$

$$F(x) = 2 + \frac{1}{12}(x-8) = \frac{1}{12}x + \frac{4}{3}.$$

the approximations are :

$F(8.05) = 2.00416667$

$F(25) = 3.41666667.$



Example 2: Determine the linear approximation for $\sin\theta$ at $\theta = 0$.

Solution : $f(\theta) = \sin\theta \qquad f'(\theta) = \cos\theta.$

$f(0) = 1 \qquad f'(0) = 1.$

the linear approximation is:

$$F(\theta) = f(0) + f'(0)(\theta - a).$$

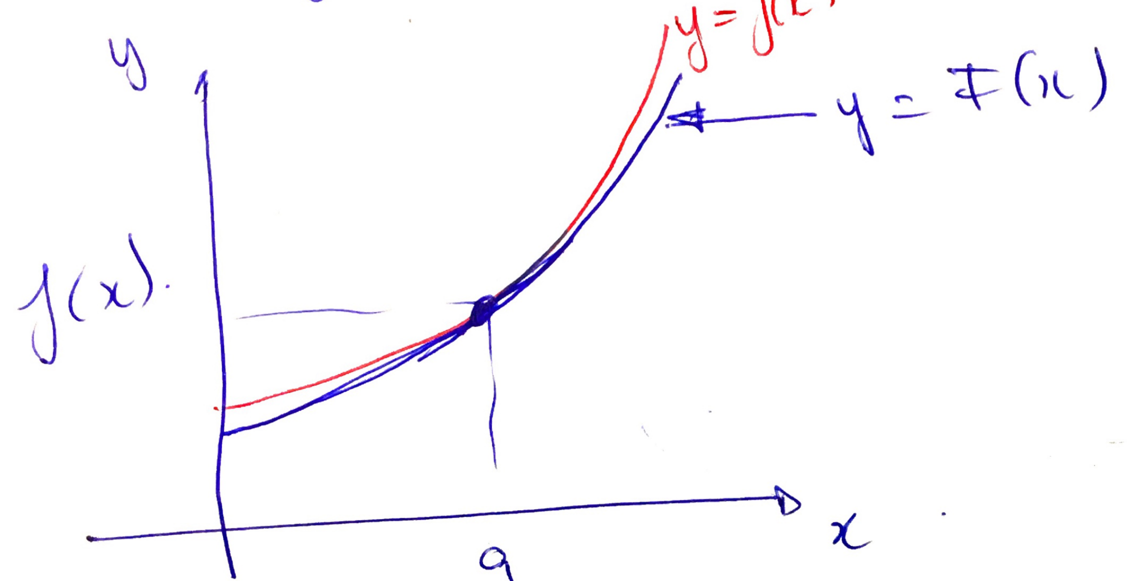
$$= 0 + 1(\theta - 0) = \theta.$$

As long as θ stays small, we can say $\sin\theta \simeq \theta$

Quadratic approximation

Given a function $f(x)$, we can find its approximation at $x = a$:

$$F(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$



We can see that near $x = a$; $F(x)$ and $f(x)$ have nearly the same graph. And in overall $F(x)$ has a better representation of an approximation of $f(x)$.

Example: Determine the quadratic approximation for $f(x) = \sqrt[3]{x}$ at $x = 8$. Use the quadratic approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$

Solution: $f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3\sqrt[3]{x^3}}$

$(x^a)^b = x^{a \cdot b}$

$x^a \cdot x^b$

$f''(x) = -\frac{2x}{9(x^2)^{4/3}}$.

$f(8) = 2$. $f'(8) = \frac{1}{12}$. $f''(8) = -\frac{1}{144}$.

$F(x) = 2 + \frac{1}{12}(x-8) + \frac{1}{2}\left(-\frac{1}{144}\right)(x-8)^2$

$\cancel{F(8.05) = 2 + 1}$

$F(x) = \frac{1}{12}x + \frac{4}{3} - \frac{1}{288}(x-8)^2$

$F(8.05) = \frac{1}{12}(8.05) + \frac{4}{3} - \frac{1}{288}(8.05-8)^2$

$\boxed{F(8.05) = 2.0041579.}$

$F(25) = \frac{1}{2}(25) + \frac{4}{3} - \frac{1}{288}(25-8)^2$

$F(25) = 3.41319$