The University of British Columbia

8 November 2017

Midterm for MATH 104

Closed book examination Time: 50 minutes

Last Name ______________________ **First** ______________

Signature ____________

Student Number ______________

Section Number: ________

Special Instructions:

No memory aids are allowed. No calculators. No communication or other electronic devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. **Where boxes are provided for answers, put your final answers in them.**

Midterms written in pencil will not be considered for regrading.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorised by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		10
5		10
Total		50

[10] **1**. **Short Answer Questions**. Each question is worth 2 points. Put your final answer in the box provided, but NO CREDIT will be given for the answer without the correct accompanying work.

(a) Find $\dfrac{d\big((1+\sin(x^2))^{-1}\big)}{dx}$.

Proposed solution. $(-1)(1+\sin(x^2))^{-2}\cos(x^2)(2x)$.

(b) Calculate y' where x and y are related via $3y^2x + 2x^2y = 5$.

Proposed solution. By implicit differentiation,

$$6yy'x + 3y^2 + 4xy + 2x^2y' = 0,$$

then $y' = -\dfrac{3y^2+4xy}{6yx+2x^2}$.

(c) Using logarithmic differentiation, find f' where $f(x) = (3x^2+2)^{\sin(x)}$.

Proposed solution. Since $\ln f(x) = \sin(x)\ln(3x^2+2)$, we get $(\ln f(x))' = \cos(x)\ln(3x^2+2) + \sin(x)\dfrac{6x}{3x^2+2}$. Since $(\ln f)' = \dfrac{f'}{f}$, we obtain $f'(x) = f(x)(\ln f(x))' = (3x^2+2)^{\sin(x)}\left(\cos(x)\ln(3x^2+2) + \sin(x)\dfrac{6x}{3x^2+2}\right)$.

(d) Find f such that $f'(x) = -3f(x)$ for all $x \in \mathbb{R}$ and $f(5) = 35$. That is, solve $\dfrac{dy}{dx} = -3y$ with $y(5) = 35$.

Proposed solution. The fundamental solution is $f(x) = ce^{-3x}$ and since $f(5) = 35$, we get $35 = ce^{-15}$ or $c = 35e^{15}$ and so, $f(x) = 35e^{15-3x}$.

(e) Find the annual interest r such that an inversion of $7,000$ dollars continuously compounded will triple in ten years.

Proposed solution. We want to find r such that $7,000e^{r10} = 21,000$ or $e^{r10} = 3$ or $r = \dfrac{\ln 3}{10}$.

[10] **2. Optimisation I.**

(a) [5] Carefully state the fundamental extreme value theorem.

A continuous function on a closed interval $[a, b]$ attains its maximum and minimum.

(b) [5] Find the maximum and the minimum of $f : [0, 7] \to \mathbb{R}$ given by $f(x) = 4x^3 - 12x + 6$.

Proposed solution. Since f is differentiable, and the maximum and minimum exist by virtue of the fundamental extreme value theorem, the maximum and minimum will be 0, 7 or any point $x^* \in (0, 7)$ such that $f'(x^*) = 0$. Now, $f'(x) = 12x^2 - 12 = 12(x^2 - 1)$ and so, $x^* = 1$ is the only point in $(0, 7)$ where the derivative of f is zero. Evaluating f in these three point: $f(0) = 6$, $f(1) = -2$ and $f(7) = 4 \times 343 - 12 \times 7 + 6 > 6$. So, -2 is minimum and $4 \times 343 - 12 \times 7 + 6$ is maximum.

[10] **3. Price Elasticity of Demand.** The price and quantity of certain commodity are related according to the expression

$$5p^2 + 9q^2 = 360.$$

(a) [4] Find the price elasticity of demand, which is given by the expression $\epsilon = \frac{p}{q}\frac{dq}{dp}$. And express it as a function of p only.

> **Proposed solution.** Implicit differentiation gives at once $2 \times 5p + 2 \times 9q \times q'(p) = 0$ or $q'(p) = -\frac{5p}{9q}$ and so, $\epsilon = -\frac{5p^2}{9q^2} = -\frac{5p^2}{360 - 5p^2}$.

(b) [2] If the manager increases price by 2% when $p = 4$, will revenue increase or decrease?

> **Proposed solution.** When $p = 4$, $|\epsilon| = \left|-\frac{5 \times 16}{360 - 5 \times 16}\right| = \frac{80}{360 - 80} = \frac{8}{28} = \frac{2}{7} < 1$ and so, a small increase in price will give a smaller decrease in quantity, that is to say, if price increases slightly the quantity will decay but so much *less* that the revenue will go up, viz., increase.

(c) [4] At what quantity will revenue be maximised?

> **Proposed solution.** This is when $|\epsilon| = 1$, that is, when $\epsilon = -1$ and so, we should solve $-\frac{5p^2}{360 - 5p^2} = -1$ or, equivalently, $\frac{5p^2}{360 - 5p^2} = 1$, which is $10p^2 = 360$, leading to $p = 6$ (since $p > 0$). Therefore, the quantity that will maximise revenue is $q > 0$ given by $5 \times 36 + 9 \times q^2 = 360$ or $q^2 = \frac{360 - 180}{9} = \frac{180}{9} = 20$ so $q = \sqrt{20}$.

[10] **4. Related Rates.** At the moment a cyclist passes directly underneath a balloon, the balloon is 20 metres above the ground. The cyclist is travelling along a straight road at a constant speed of 12 metre per second, and the balloon is rising at a constant rate of 3 metres per second. How fast is the distance between the cyclist and the balloon changing 5 seconds after the balloon is directly above the cyclist? How many seconds after the cyclist was right underneath the balloon will the two be at 65 metres away?

Proposed solution. By Pythagoras theorem, the distance between the two is given by $a^2 = b^2 + c^2$, where b is the distance from the balloon to the ground and c is the distance from the point where the cyclist was directly underneath the balloon until now. Then, $\frac{db}{dt} = 3$ and $\frac{dc}{dt} = 12$. Then, implicit differentiation gives $2a\frac{da}{dt} = 6b + 24c$ or $\frac{da}{dt} = \frac{3b + 12c}{a}$. When $t = 5$, $b = 20 + 15 = 35$ and $c = 60$, so $a = \sqrt{35^2 + 60^2}$ and then

$$\frac{da}{dt} = \frac{105 + 720}{\sqrt{35^2 + 60^2}} = \frac{825}{\sqrt{35^2 + 60^2}}.$$

For the second question, we are given $a = 65$, so $b^2 + c^2 = 65^2$ or, equivalently $(20 + 3t)^2 + (12t)^2 = 65^2$ and so, $400 + 120t + 9t^2 + 144t^2 = 4,225$ or $153t^2 + 120t - 3825 = 0$ and so (since $t > 0$)

$$t = \frac{-120 + \sqrt{120^2 + 4(153)(3825)}}{306}.$$

[10] **5. Optimisation II.** A furniture store expects to sell 640 sofas at a steady rate next year. The manager of the store plans to order these sofas from the manufacturer by placing several orders of the same size spaced equally throughout the year. The ordering cost for each delivery is \$160, and the carrying costs amount to \$32 per sofa per delivery. Determine how many sofas per order should the manager request in order to minimise cost.

(a) [3] Define the objective function.

Proposed solution. Let s be the number of sofas per delivery and d the number of deliveries. We got $sd = 640$, so $s = \frac{640}{d}$. The total cost is $160d + 32s = 160d + \frac{32 \times 640}{d}$. The objective function is $C = 160d + \frac{32 \times 640}{d}$. The domain of C is $d > 0$.

(b) [3] Find possible minimisers.

Proposed solution. Since C is differentiable everywhere, any minimiser d^* will satisfy $C'(d^*) = 0$ and so, we should solve $C'(d) = 0$. Now, $C' = 160 - \frac{32 \times 640}{d^2}$ and so $C' = 0$ yields $d^2 = \frac{32 \times 640}{160} = 32 \times 6416 = 2 \times 64 = 128$. Then $d^* = \sqrt{128} = \sqrt{2^7} = 2^3\sqrt{2} = 8\sqrt{2}$ (since $d > 0$) is the only possible minimiser.

(c) [4] Find global minimiser.

Proposed solution. The second derivative of C is $C'' = \frac{2 \times 32 \times 640}{d^3} > 0$ everywhere. Therefore, d^* is global minimiser. This showed that $s^* = \frac{640}{d^*} = \frac{640}{8\sqrt{2}} = \frac{80}{\sqrt{2}} = 40\sqrt{2}$. Since d^* and s^* are not integers, the manager should place an order with integers closest to d^* and s^*.