Asymptotes

An assymptote is a line that a graph approaches, but not intersect.

- vertical asymptotes
- horizontal asymptotes
- oblique (slant) asymptotes.

- **Vertical asymptotes**

Method 1: Definition of vertical asymptote

the line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following is true:

$$\lim_{x \to a} f(x) = \pm\infty$$

$$\lim_{x \to a^-} f(x) = \pm\infty$$

$$\lim_{x \to a^+} f(x) = \pm\infty$$

Method 2:

For rational functions, vertical asymptotes are vertical lines that correspond to the zeroes of the denominator

Given the rational function $f(x)$.

step 1: Write $f(x)$ in reduced form.

step 2: if $x - c$ is a factor in the denominator then $x = c$ is the vertical asymptote

Example 1: Find the vertical asymptote of

$$f(x) = \frac{4x}{x-3}$$

Solution:

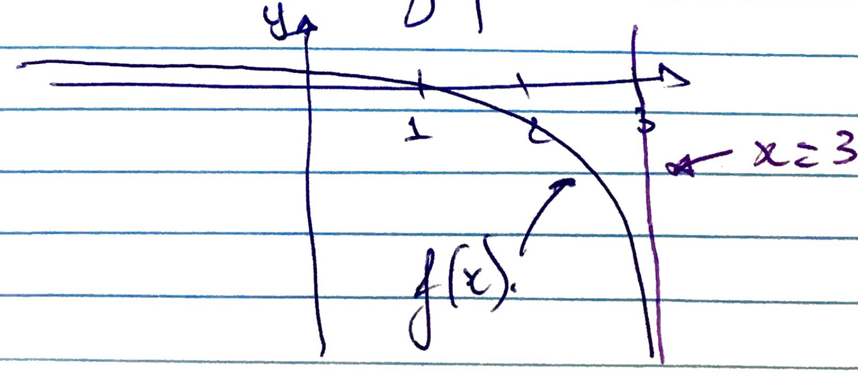
* Method 1: $\lim_{x \to 3^+} \frac{4x}{x-3} = \infty$

The line $x = 3$ is a vertical asymptote.

* Method 2:

step 1: $f(x)$ is already reduced form

step 2: the denominator is $x - 3$ and so the vertical asymptote is $x = 3$.



- Horizontal asymptote

* Method 1: Definition of horizontal asymptote

The line $y=L$ is called a horizontal asymptote of the curve $y=f(x)$ if either

$$\lim_{x\to\infty} f(x)=L \quad \text{or} \quad \lim_{x\to-\infty} f(x)=L$$

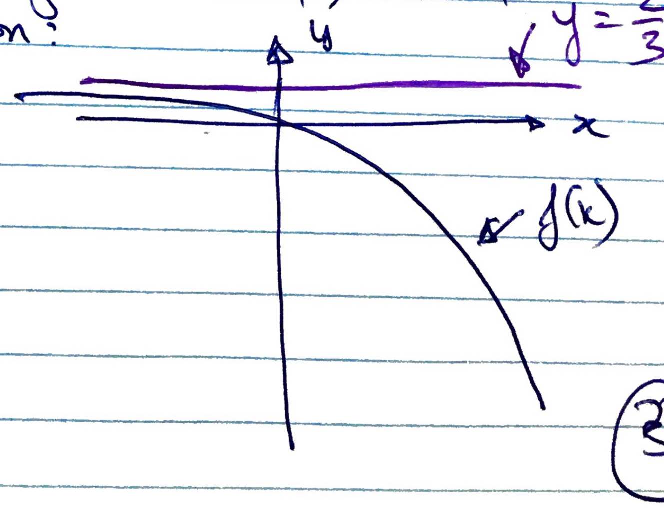
* Method 2: For the rational function $f(x)$

If the degree of x ~~in~~ the numerator is less than the degree of x in the denominator then $y=0$ is the horizontal asymptote.

If the degree of x in the numerator is equal to the degree of x in the denominator then $y=c$ where c is obtained by dividing the leading coefficients.

Example: Find the horizontal and vertical asymptotes of the function:

$$f(x)=\frac{2x+1}{3x-5}$$



Solution:

Method 1:

$$\lim_{x\to\infty} \frac{2x+1}{3x-5} = \lim_{x\to\infty} \frac{2+\cancel{\frac{1}{x}}^{\to 0}}{3-\cancel{\frac{5}{x}}^{\to 0}} = \frac{2}{3}.$$

$y = \frac{2}{3}$ is the horizontal asymptote.

<u>Method 2:</u> the degree of x in the numerator is equal to the degree of x in the denominator.

$\to$ Dividing the leading coefficients, we get $\frac{2}{3}$

$y = \frac{2}{3}$ is the horizontal asymptote

- <u>Oblique asymptote</u> (slant asymptote)

If $\lim_{x\to\infty} [f(x) - (mx+b)] = 0$, then line $y = mx+b$ is called the oblique (slant) asymptote because the vertical distances between the curve $y = f(x)$ and the line $y = mx+b$ approaches 0

For rational functions, oblique asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case, the equation of the oblique asymptote can be found by long division

Example: Find the asymptotes of the function

$$f(x) = \frac{x^3}{x^2+1}$$

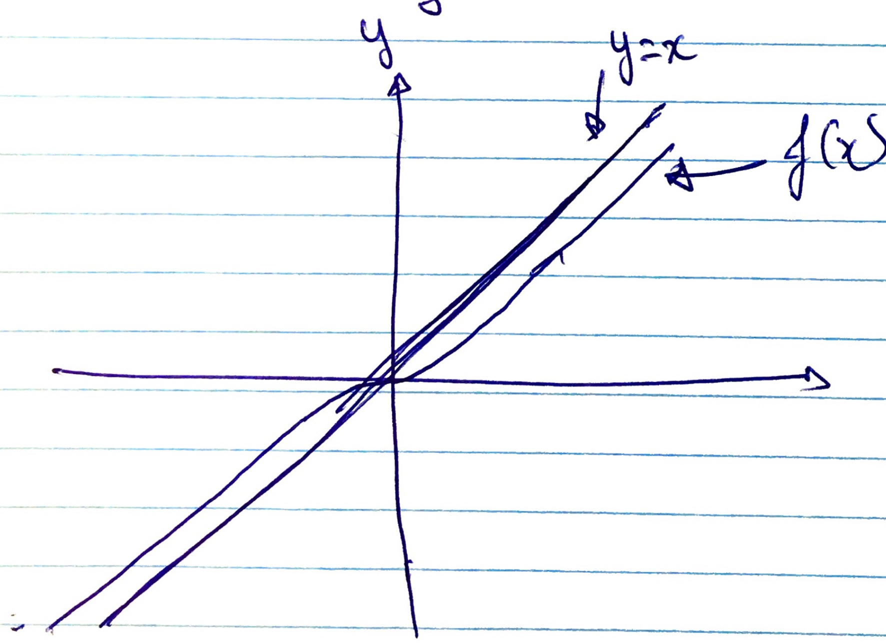
$x^2+1 > 0 \Rightarrow$ no vertical asymptote.

$$\begin{array}{r|l} x^3 & x^2+1 \\ \hline -(x^3+x) & x \\ \hline -x & \end{array}$$

(larg dicision)

$$f(x) = \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

the line $y = x$ is the oblique asymptote.



Example: Find the asymptotes of.

$$f(x) = \frac{x^2+1}{x-4}$$

$$\begin{array}{r|l} x^2+1 & x-4 \\ -\ \underline{x^2-4x} & x+4 \\ 4x+1 & \\ -\ \underline{4x-16} & \\ 17 & \end{array}$$

$$f(x) = \frac{x^2+1}{x^2-4x} = x+4+\frac{17}{x-4}$$

$y = x+4$ is the oblique asymptote.