Taylor Polynomials

It is a better approximation. It uses polynomial functions of any degree we like. As we saw previously, we determine the coefficients of the polynomial by requiring, that at the point $x=a$, the approximation and its first n derivatives agree with those of the original function.

Given a function $f(x)$, its approximation is $T_n(x)$

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

$$= \sum_{k=0}^{n} c_k (x-a)^k$$

Now consider $T_n(x)$ and its derivatives:

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

$$T_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1}$$

$$T''(x) = + 2c_2 + 6c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$

$$T'''(x) = 6c_3 + \cdots + n(n-1)(n-2)c_n(x-a)^{n-3}$$

$$T_n^{(n)}(x) = n!\, c_n$$

When we substitute $x=a$ into the previous expressions only the constant Terms survive and we get.

$$T_n(a)=c_0 \qquad T_n''(a)=2c_2 \qquad T_n^{(n)}(a)=n!\,c_n.$$
$$T_n'(a)=c_1 \qquad T_n'''(a)=6c_3$$

So now if we set the coefficients of $T_n(x)$ so that it agrees with $f(x)$ at $x=a$, then we need.

$$T_n(a)=c_0=f(a) \Rightarrow c_0=f(a)=\frac{1}{0!}f(a).$$
$$T_n'(a)=c_1=f'(a) \Rightarrow c_1=f'(a)=\frac{1}{1!}f'(a).$$
$$T_n''(a)=2c_2=f''(a) \Rightarrow c_2=\frac{1}{2}f''(a)=\frac{1}{2!}f''(a)$$
$$T_n'''(a)=6c_2=f'''(a) \Rightarrow c_3=\frac{1}{6}f'''(a)=\frac{1}{3!}f'''(a)$$

More generally.

$$T_n^{(k)}(a)=k!\,c_k=f^{(k)}(a) \Rightarrow c_k=\frac{1}{k!}f^{(k)}(a).$$

And finally the n^{th} derivative is

$$T_n^{(n)}(a)=n!\,c_n=f^{(n)}a \Rightarrow c_n=\frac{1}{n!}f^{(n)}(a)$$

Putting this all together gives:

$$\boxed{\begin{aligned} f(x)\simeq T_n(x) &= f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\dots+\frac{1}{n!}f^{(n)}(a)(x-a)^n \\ &= \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(x-a)^k \end{aligned}}$$

Example: $f(x) = e^{x}$.

The constant, linear and quadratic approximations are:

$T_0(x) = 1 \qquad T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2}$.

$f'(x) = e^{x} \qquad f''(x) = e^{x} \qquad f^{(n)}(x) = e^{x}$.

The approximation of $f(x)$ at $a = 0$ is:

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$$

$$T_7(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040}.$$

Thus, the approximate value of e^1 is:

$$e^1 \simeq T_7(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$$

$$= \frac{685}{252} = 2.718253968\ldots$$

Note that $e^1 = 2.78281828\ldots$

Example: Compute the 5^{th} Taylor polynomial for $\log x$ about $x = 1$.

Solution: $a = 1$.

$f(x) = \log x \qquad f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\qquad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(1) = 2$

$f^{(4)}(x) = -\frac{6}{x^4}$ $\qquad f^{(4)}(1) = -6$

$f^{(5)}(x) = \frac{24}{x^5}$ $\qquad f^{(5)}(1) = 24$

Using the "Taylor equation"

$$T_5(x) = 0 + 1\cdot(x-1) + \frac{1}{2}(-1)(x-1)^2 + \frac{1}{6}\,2(x-1)^3 + \frac{1}{24}(-6)(x-1)^4 + \frac{1}{120}(x-1)^5$$

$$T_5(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$

With a little work, we can write:

$$T_5(x) = \sum_{k=1}^{5} \frac{(-1)^{k+1}}{k}(x-1)^k$$

Example: Find the 4^{th} Taylor Polynomial for $\cos x$ about $x=0$.

Solution: We have $a=0$ and we need to find the first 4 derivatives of $\cos x$.

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$

$\Rightarrow T_4(x) = 1 + 1(0)\,x + \frac{1}{2}(-1)\,x^2 + \frac{1}{6}(0)\,x^3 + \frac{1}{24}(1)\,x^4$

$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$