# Math 104 section 108 Homework week 7 solutions

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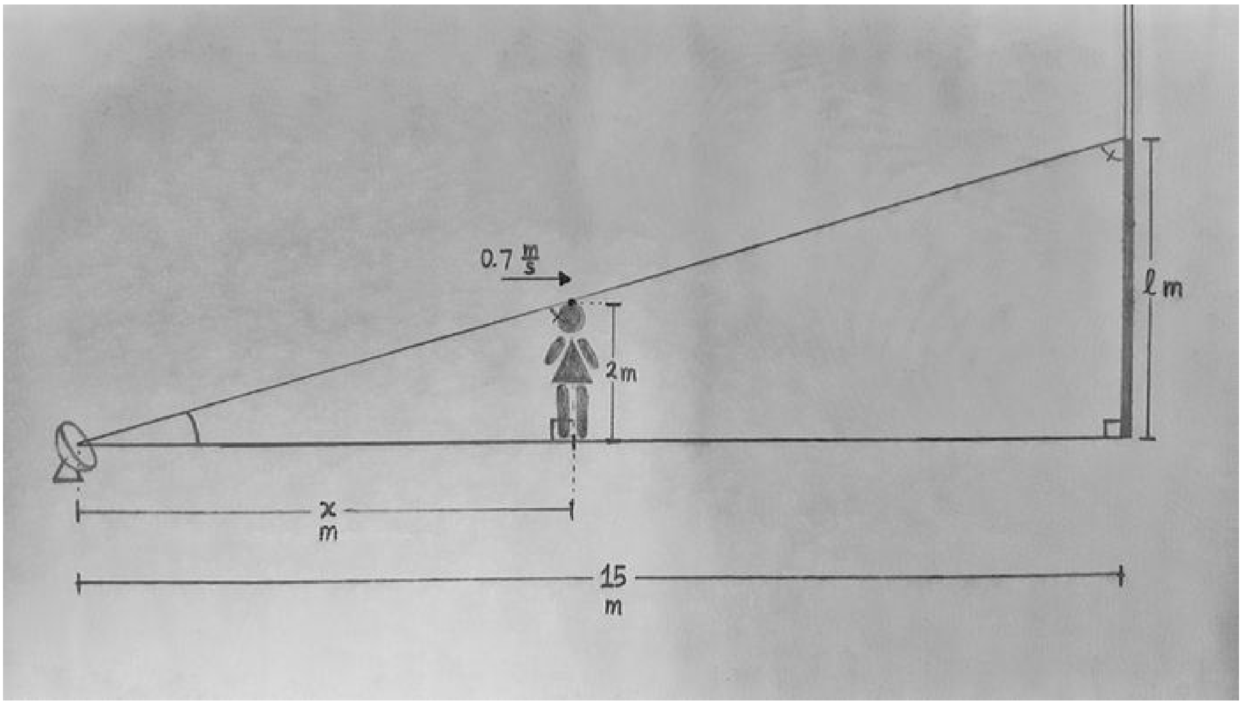
**Exercise 0.1.** A spotlight on the ground shines on a wall $15m$ away. If a woman $2m$ tall walks from the spotlight toward the wall at a speed of $0.7m/s$, how fast is the length of her shadow (on the building) changing when she is $8m$ from the building? State your answer accurate to 2 decimal places. (1.5 marks)

Let be $x$ the distance of the woman from the spotlight and let be the length of her shadow on the wall (both in metres). Note that the distance from the spotlight to the wall (15 m) and the height of the woman (2 m) remain constant throughout the problem.




The quantity $\frac{dx}{dt}$ (where represents time) therefore represents the rate of change in the distance of the woman from the spotlight, i.e., her speed. As the woman walks towards the wall, her distance from the spotlight increases; hence $\frac{dx}{dt} = +0.7$ (metres per second). The quantity that we seek is $\frac{dl}{dt}$ the rate of change in the length of her shadow.

Next, we observe that there are two similar right-angled triangles in the problem.

So the ratios of their leg lengths are equal, i.e., $\frac{l}{15} = \frac{2}{x}$. Rearranging this, we obtain $l = \frac{30}{x}$. (0.5 mark)

We now have an equation relating $l$ to $x$ which we can differentiate with respect to time to find an equation relating $\frac{dl}{dt}$(which we seek) to $\frac{dx}{dt}$ (which is known).

$$\frac{dl}{dt} = \frac{d}{dt}\left(\frac{30}{x}\right) = -\frac{30}{x^2} \cdot \frac{dx}{dt} \quad \text{(0.5 mark)}$$

At the instant in the question, the woman is 8 m from the wall, so she is $x = 15 - 8 = 7$ metres from the spotlight. We now simply substitute in all the information we know:

$$\frac{dl}{dt} = -\frac{30}{7^2} \cdot 0.7 = -\frac{21}{49} = -\frac{3}{7} \simeq -0.43 \quad \text{(0.5 mark)}$$

Next, we observe that there are two similar right-angled triangles in the problem (outlined below):

**Exercise 0.2.** You borrow 10 thousand dollars from Nick the Shark, who charges you at a fixed rate $r$ that is compounded continuously. If you pay Nick 100 thousand dollars 2 years later, what was the annual rate of interest that he charged? (A calculator-ready form will suffice.) (1.5 marks)

For continuously compounded interest, we have the formula $A = Pe^{rt}$ where $A$ is the amount, $P$ is the principal, $r$ is the rate, and $t$ is the time. We are given that $P = 10000, A = 100000, t = 2$. Plugging these in yields:

$$\begin{aligned} 100000 &= 10000e^{2r} \\ 10 &= e^{2r} \\ r &= \frac{\ln 10}{2} \end{aligned}$$

**Exercise 0.3.** Let $f(x) = \frac{e^x}{x^2}$ (4 marks)

1. Find the critical point of $f(x)$.

First, we determine $f'(x)$. Rewrite $f(x) = e^x \cdot x^{-2}$

$$\begin{aligned} f'(x) &= e^x x^{-2} - 2x^{-3}e^x \\ &= \frac{e^x}{x^2} - 2\frac{e^x}{x^3} \\ &= \frac{e^x(x-2)}{x^3} \quad \text{(0.5 mark)} \end{aligned}$$

The critical points of $f(x)$ are at $x$ where $f'(x) = 0$.

$$\begin{aligned} \frac{e^x(x-2)}{x^3} &= 0 \\ x - 2 &= 0 \\ x &= 2 \quad \text{(0.5 mark)} \end{aligned}$$

2. Find the intervals on which $f$ is increasing or decreasing. (1 mark)

| | $(-\infty, 0)$ | $(0, 2)$ | $(2, \infty)$ |
|---|---|---|---|
| $e^x$ | $+$ | $+$ | $+$ |
| $(x-2)$ | $-$ | $-$ | $+$ |
| $x^3$ | $-$ | $+$ | $+$ |
| $f'$ | $+$ | $-$ | $+$ |
| $f$ | increasing | decreasing | increasing |

3. Find $f''(x)$. (1 mark)

$$\begin{aligned} f''(x) &= \frac{e^x(x-1)x^3 - 3e^x(x-2)x^2}{x^6} \\ &= \frac{e^x x^2[x(x-1) - 3(x-2)]}{x^6} \\ &= \frac{e^x(x^2 - 4x + 6)}{x^4} \end{aligned}$$

4. Find the nature of the critical points. (1 mark)

$f''(2) > 0 \Longrightarrow$ relative minimum.

**Exercise 0.4.** Opad, the blockbuster product of Opple Inc., has a weekly demand $q$ that declines with price $p$ according to $q = 1000e^{-p/200}$. (3 marks)

1. Find the elasticity of demand $\varepsilon$ at the current price of \$100. (1 mark)

$$\begin{aligned} \varepsilon(p) &= \frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{1000e^{-p/200}} 5e^{-p/200} = \frac{p}{200} \\ \varepsilon(100) &= \frac{1}{2} < 1 \Longrightarrow \text{price inelastic} \end{aligned}$$

2. Use the elasticity of demand to calculate the marginal revenue at the current price of \$100. Simplify your answer to "calculator ready". (1 mark)

$$MR = p \cdot \left(1 + \frac{1}{\varepsilon}\right) = 100 \cdot \left(1 + \frac{1}{0.5}\right) = 300$$

3. If the price is raised by 1%, use the elasticity to estimate the percentage decline in sales. (1 mark)

$$0.01 \times 0.5 = 0.005 = 0.5\%$$