[15] **1. Short Answer Questions**. Each question is worth 3 points. Put your final answer in the box provided, but NO CREDIT will be given for the answer without the correct accompanying work.

(a) Evaluate $\lim_{x\to\infty} \dfrac{3x^2-1}{x^2+\sqrt{x+1}}$.

$$= \lim_{x\to\infty} \frac{x^2\left(3-\cancel{\frac{1}{x^2}}\right)}{x^2\left(1+\sqrt{\cancel{\frac{x}{x^4}}+\cancel{\frac{1}{x^4}}}\right)} = 3$$

Answer: 3

(b) Calculate $f'(x)$ where $f(x) = \dfrac{e^x - 3x}{1+x}$.

$$f'(x) = \frac{(e^x-3)(1+x)-(e^x-3x)}{(1+x^2)^2}$$

$$= \frac{xe^x-3}{(1+x)^2}$$

Answer: $\dfrac{xe^x-3}{(1+x)^2}$

(c) Evaluate $\lim_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$.

$$\frac{\sqrt{x^2+8}-3}{x+1} = \frac{\sqrt{x^2+8}-3}{x+1}\cdot\frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \frac{x^2+8-9}{(x+1)(\sqrt{x^2+8}+3)} = \frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)}$$

$$= \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}(\sqrt{x^2+8}+3)} = \frac{x-1}{\sqrt{x^2+8}+3}.$$

$$\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x\to -1}\frac{x-1}{\sqrt{x^2+8}+3} = -\frac{2}{6} = -\frac{1}{3}$$

Answer: $-1/3$

(d) Let $f(x) = \sqrt{1-x^2}$ for $-1 \le x \le 1$. Find the equation of the tangent line to $f(x)$ at $x = \sqrt{3}/2$.

$$f'(x) = -\frac{2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}.$$

Answer: $y = -\sqrt{3}x + 2$.

$$f'\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}\left(\frac{1}{\left(1-\frac{3}{4}\right)^{1/2}}\right) = -\frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{1/4}}\right) = -\sqrt{3}$$

$$\frac{y-y_0}{x-x_0} = m = -\sqrt{3} \Rightarrow y - y_0 = -\sqrt{3}(x-x_0) \Rightarrow y_0 = \sqrt{1-\left(\frac{3}{4}\right)} = \frac{1}{2}$$

$$y = -\sqrt{3}\cdot(x-x_0) + \frac{1}{2} = -\sqrt{3}x + \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}.$$

(e) Find the values of a and b so that $f(x) = \begin{cases} 2+3e^x & \text{if } x < 0, \\ ax+b & \text{if } x \ge 0 \end{cases}$ is differentiable everywhere.

$$\lim_{x\to 0^-} f(x) = \lim_{x \to 0^-} 2+3e^x = 2+3e^0 = 5$$

Answer: $a = 3$; $b = 5$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} ax+b = b \Rightarrow \boxed{b=5}$$

$$\left.\frac{df}{dx}\right|_{x=0^-} = \left.3e^x\right|_{0^-} = 3; \quad \left.\frac{df}{dx}\right|_{x=0^+} = \left.a\right|_{0^+} = a.$$

$$\Rightarrow \boxed{a=3}$$

[10] **2. Definition of the Derivative.**

(a) [3] Carefully state the definition of the derivative of a function $f(x)$ at a point $x = a$.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When the limit exists, the function is said to be differentiable.

(b) [5] Use the definition of the derivative to compute the derivative of $f(x) = \dfrac{x}{x-1}$ at $x = 2$. NO CREDIT will be given for using any other method.

$$f'(2) = \lim_{x \to 2} \frac{\frac{x}{x-1} - 2}{x - 2}$$

$$= \lim_{x \to 2} \frac{x - 2(x-1)}{(x-1)(x-2)} = \frac{x - 2x + 2}{(x-1)(x-2)}$$

$$f'(2) = \lim_{x \to 2} \frac{-x+2}{(x-1)(x-2)} = \lim_{x \to 2} \frac{-\cancel{(x-2)}}{(x-1)\cancel{(x-2)}}$$

$$\boxed{f'(2) = -1}$$

[10] **3.** The cost of producing q tons of sugar is given by $C(q) = \frac{q^2}{20} + 9q + 120$. When 10 tons of sugar are sold, the price at which the producer can sell them is \$269. For every extra 10 tons of sugar the producer sells in the market, the price drops by \$1 per ton.

(a) [2] Find the linear demand equation for sugar. Use p for the unit price and q for the weekly demand.

$p = mq + b$.

$p = -\frac{1}{10}q + b$. $(10, 269) \Rightarrow 269 = -\frac{1}{10}(10) + b$.

$\Rightarrow b = 270$

$$\boxed{p = -\frac{1}{10}q + 270}$$

(b) [2] Find the weekly revenue function $R(q)$ *as a function of* q.

$R = p \cdot q$

$$\boxed{R = -\frac{1}{10}q^2 + 270q}$$

(c) [2] Find the break-even points for the sugar producer. Give both the price p and quantity q at each of these points.

$R(q) = C(q)$

$-\frac{1}{10}q^2 + 270q = \frac{q^2}{20} + 9q + 120$

$\frac{3q^2}{20} - 261q + 120 = 0 \Rightarrow q^2 - 1740q + 800 = 0.$

$\Delta = 7561 \quad q_1 = 0.459 \quad q_2 = 1739.5$

$p_1 = 356.95 \quad p_2 = 96.046.$

(d) [2] Find the derivative of the profit function, $P'(q)$. (This derivative is usually called the *marginal profit.*)

$P = R - C. = -\frac{1}{10}q^2 + 270q - \left(\frac{q^2}{20} + 9q + 120\right)$

$P = -\frac{3}{20}q^2 + 261q - 120$

$\boxed{\frac{dp}{dq} = -\frac{3}{10}q + 261}$

(e) [2] Suppose that the producer is producing and selling $\hat{q}$ tons of sugar, where $\hat{q}$ corresponds to the largest q-value of all the break-even points. Should the producer increase or decrease the amount of sugar it is producing to increase its profit? Explain your answer.

$\tilde{q} = 1739.5$

$P(\tilde{q}) = -\frac{3}{20}(1739.5)^2 + 261(1739.5) - 120$

$P(\tilde{q}) = 10.4625$

$\frac{P(870) - P(1739.5)}{\cancel{P}870 - 1739.5} < 0 \Rightarrow$ decrease.

[15] **4.** Consider the function $f(x) = 1 - 2x - x^2$. Find the points (x_0, y_0) and (x_1, y_1) such that the tangent lines to $f(x)$ at (x_0, y_0) and (x_1, y_1) are parallel and orthogonal, respectively, to the given line $y = x + 10$.

$f'(x) = -2-2x$.

$\frac{y-y_0}{x-x_0} = -2-2x_0 \Rightarrow y_0 = 1-2\left(-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)^2$

$y_0 = 2-\frac{1}{4} = \frac{7}{4}$

* Orthogonal (perpendicular)

$-2-2x_0 = -1$

$\boxed{x_0 = -\frac{1}{2}} \Rightarrow \boxed{y_0 = \frac{7}{4}}$

* Parallel.

$y_0 = 1-2\left(-\frac{3}{2}\right)-\left(-\frac{3}{2}\right)^2$

$= 4-\frac{9}{4} = \frac{7}{4}.$

$-2-2x_0 = 1$

$\boxed{x_0 = -\frac{3}{2}} \Rightarrow \boxed{y_0 = \frac{7}{4}}$