Midterm 1 MATH 104 section 108

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Short Problems

1. Find the value of a and b such that the function $f(x)$ is continuous everywhere.

$$f(x) = \begin{cases} ax + b & x < 0 \\ \dfrac{2\cos(x)}{1 + 2\cos(x) + \sin(x)} & x \geq 0 \end{cases}$$

Solution:

(a) The continuity of the function $f(x)$ need to be investigated at $x = 0$. i.e.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\lim_{x \to 0^-} ax + b = b$$
$$\lim_{x \to 0^+} \frac{2\cos(0)}{1 + 2\cos(0) + \sin(0)} = \frac{2}{3}$$
$$f(0) = b$$

It gives $b = \dfrac{2}{3}$.

(b) In order for the derivative $f'(x)$ to exist at $x = 0$, the limit $\lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$ must exist. In particular the limits $\lim_{h \to 0^-} \dfrac{f(h) - f(0)}{h}$ and $\lim_{h \to 0^+} \dfrac{f(h) - f(0)}{h}$ must exist and equal to each other.

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{ah + \frac{2}{3} - \frac{2}{3}}{h} = \frac{d}{dx}\left[ax + \frac{2}{3}\right]_{x=0} = a$$
$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \frac{d}{dx}\left[\frac{2\cos(x)}{1 + 2\cos(x) + \sin(x)}\right]_{x=0} = -\frac{-2\sin(0) + \cos(0)}{(1 + 2\cos(0) - \sin(0))^2} = -\frac{2}{9}$$

It gives $a = -\dfrac{2}{9}$

The answer is

$$f(x) = \begin{cases} -\dfrac{2}{9}x + \dfrac{2}{3} & x < 0 \\ \dfrac{2\cos(x)}{1 + 2\cos(x) + \sin(x)} & x \geq 0 \end{cases}$$

2. Evaluate $\displaystyle\lim_{x \to -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

Solution:

$$\begin{aligned}
\frac{\sqrt{x^2+8}-3}{x+1} &= \frac{\sqrt{x^2+8}-3}{x+1} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3} \\
&= \frac{(x^2+8)-3^2}{(x+1)\sqrt{x^2+8}+3} \\
&= \frac{x^2-1}{(x+1)\sqrt{x^2+8}+3} \\
&= \frac{x-1}{\sqrt{x^2+8}+3} \\
\lim_{x\to-1} \frac{\sqrt{x^2+8}-3}{x+1} &= \lim_{x\to-1} \frac{x-1}{\sqrt{x^2+8}+3} = -\frac{1}{3}
\end{aligned}$$

3. Evaluate $\lim_{t\to0} \frac{(t+4)^2-16}{(t-5)^2-25}$

Solution:

$$\begin{aligned}
\frac{(t+4)^2-16}{(t-5)^2-25} &= \frac{(t+4)-4}{(t-5)-5} \cdot \frac{(t+4)+4}{(t-5)+5} \\
&= \frac{t}{t-10} \cdot \frac{t+8}{t} \\
&= \frac{t+8}{t-10} \\
\lim_{t\to0} \frac{(t+4)^2-16}{(t-5)^2-25} &= \lim_{t\to0} \frac{t+8}{t-10} = -\frac{4}{5}
\end{aligned}$$

4. Find the derivative of the following function using the definition of the derivative:

$$f(x) = 2x^2 - 10x + 5$$

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to0} \frac{2(x+h)^2 - 10(x+h) + 5 - (2x^2 - 10x + 5)}{h} \\
&= \lim_{h\to0} \frac{2x^2 + 4xh + 2h^2 - 10x - 10h + 5 - 2x^2 + 10x - 5}{h} \\
&= \lim_{h\to0} \frac{4xh + 2h^2 - 10h}{h} \\
f'(x) &= 4x - 10
\end{aligned}$$

5. Find the derivative of the following function using the definition of the derivative:

$$f(x) = \frac{x}{x-1}$$

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(x+h)(x-1)}{(x+h-1)(x-1)} - \frac{x(x+h-1)}{(x+h-1)(x-1)}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{x^2 - x + xh - h - x^2 - xh + x}{(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \\
f'(x) &= -\frac{1}{(x-1)^2}
\end{aligned}
$$

6. Differentiate the following functions:

   (a) $f(x) = \sqrt[3]{\ln x + 2x}$

   Solution:

   $$\frac{df}{dx} = \frac{1}{3}\left(\frac{1}{x}+2\right)(\ln x + 2)^{-\frac{2}{3}}$$

   (b) $h(x) = \dfrac{x^2}{e^x + x}$

   Solution:

   $$
   \begin{aligned}
   \frac{dh}{dx} &= \frac{2x(e^x+x) - (e^x+1)x^2}{(e^x+x)^2} \\
   &= \frac{2xe^x + 2x^2 - x^2e^x - x^2}{(e^x+x)^2} \\
   &= \frac{2xe^x + x^2 - x^2e^x}{(e^x+x)^2} \\
   \frac{dh}{dx} &= \frac{xe^x(2-x) + x^2}{(e^x+x)^2}
   \end{aligned}
   $$

7. Find the equation of the tangent line with a slope $m = \dfrac{1}{2}$ for the curve $y = f(x) = \sqrt{1-x^2}$ for x in $[-1, 1]$.

   Solution:

   - The derivative at any point x is $f'(x) = -\dfrac{x}{\sqrt{1-x^2}}$
   - The slope of the tangent line to $f(x)$ at $x = x_0$ is $f'(x_0) = -\dfrac{x_0}{\sqrt{1-x_0^2}}$

   $$f'(x_0) = -\frac{x_0}{\sqrt{1-x_0^2}} = \frac{1}{2}$$

It gives:

$$x_0^2 = \frac{1}{4}(1 - x_0^2)$$
$$x_0^2(1 + \frac{1}{4}) = \frac{1}{4}$$
$$x_0^2 = \frac{1}{5} \Longrightarrow x_0^2 = \pm\sqrt{\frac{1}{5}} = \pm\frac{\sqrt{5}}{5}$$

Plugging x_0 into $f(x)$, $y_0 = \sqrt{1 - \frac{1}{5}} = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5}$.

- Knowing that the equation of a tangent line is $y = mx+b$. For this case $m = \frac{1}{2}$ and the equation of the tangent line as the form $y = \frac{1}{2}x + b$.
- Plugging (x_0, y_0) into the that equation gives:

$$\frac{2\sqrt{5}}{5} = \frac{1}{2} \cdot \frac{-\sqrt{5}}{5} + b$$

Here, we take $x_0 < 0$ for a positive slope. Therefore,

$$b = \frac{2\sqrt{5}}{5} + \frac{1}{2} \cdot \frac{\sqrt{5}}{5} = \frac{\sqrt{5}}{2}$$

The final answer is $y = \frac{1}{2}x + \frac{\sqrt{5}}{2}$

Long Problem

Rowlex is a producer of high quality watches. Let p be the price of a watch (in dollars) and let q be the weekly demand of watches in Vancouver. The owner estimates that the price and demand are related by the following equation:

$$50p + 20q = 8000$$

1. If the Rowlex watch costs 100$, how many watches will be sold per week?

Solution:

$$50(100) + 20q = 8000$$
$$20q = 8000 - 5000$$
$$q = \frac{3000}{20}$$
$$q = 150$$

2. Express the price p in terms of the demand q.

Solution:

$$50p = 8000 - 20q$$
$$p = \frac{8000 - 20q}{50}$$
$$p = 160 - \frac{2q}{5}$$

3. Express the revenue R as a function of q.

Solution:
The revenue is given by $R = p \cdot q$

$$R = q \cdot (160 - \frac{2q}{5})$$

4. The weekly production cost is 500$ to keep Rowlex running plus a variable cost of 10$ per unit watch. Express the Rowlex's weekly profit (P) as a function of q.

Solution:
The profit is given by $P = R(q) - C(q)$

$$P = q \cdot (160 - \frac{2q}{5}) - (500 + 10q)$$
$$P = -\frac{2q^2}{5} + 150q - 500$$

5. How many watches should Rowlex produce in order to maximize its profit? What should be the price of a watch in this case?

Solution:
The maximum profit is given by $\frac{dP}{dq} = 0$

$$\frac{dP}{dq} = \frac{d}{dq}\left(-\frac{2q^2}{5} + 150q - 500\right) = 0$$
$$= -\frac{4q}{5} + 150q = 0 \Longrightarrow q = \frac{150 \times 5}{4} = 187.5$$

Rowlew should sell 187 watches in order to maximize its profit.
The price of a watch is :

$$p = 160 - \frac{2(187)}{5} = 85.2\$$$