Derivatives of Inverse trig functions

If $f(x)$ and $g(x)$ are inverse functions then

$$g'(x) = \frac{1}{f'[g(x)]}$$

Two functions are inverses if $f[g(x)] = x$ and $g[f(x)] = x$

Inverse of sine:

$y = \sin^{-1} x \Rightarrow x = \sin(y)$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

$-1 \leq \sin(y) \leq 1 \Rightarrow -1 \leq x \leq 1$

Example: Evaluate $\sin^{-1}\left(\frac{1}{2}\right)$

$$\sin(y) = \frac{1}{2}$$

From a unit circle, we have $y = \frac{\pi}{6}$

We have the following relationship between the inverse sine function and the sine function.

$\sin(\sin^{-1} x) = x$. $\sin^{-1}(\sin x) = x$.

They are inverses of each other.

$f(x) = \sin(x) \quad g(x) = \sin^{-1}(x)$

then $g'(x) = \dfrac{1}{f'[g(x)]} = \dfrac{1}{\cos[\sin^{-1}(x)]}$

It's not a very useful formula.

$y = \sin^{-1}(x) \Rightarrow x = \sin(y)$.

$\cos(\sin^{-1}(x)) = \cos(y)$.

$\cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \sqrt{1-\sin^2 y}$.

$\cos(\sin^{-1} x) = \cos y = \sqrt{1-\underbrace{\sin^2 y}_{x}} = \sqrt{1-x^2}$

$$\boxed{g'(x) = \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}}$$

* Inverse of cosine

$y = \cos^{-1} x \Rightarrow \cos y = x$ for $0 \leq y \leq \pi$

$-1 \leq x \leq 1$. because $-1 \leq \cos y \leq 1$.

Example : Evaluate $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$

We are asking the following. $\cos y = -\frac{\sqrt{2}}{2}$.

The unit circle give $y = \frac{3\pi}{4}$, The inverses cosine and cosine functions are inverses of each other.

$$\cos(\cos^{-1} x) = x \qquad \cos^{-1}(\cos x) = x .$$

$$f(x) = \cos x \qquad g(x) = \cos^{-1} x .$$

$$g'(x) = \frac{1}{f'[g(x)]} = \frac{1}{-\sin(\cos^{-1} x)} .$$

Upon simplifying, we get:

$$\boxed{\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}}$$

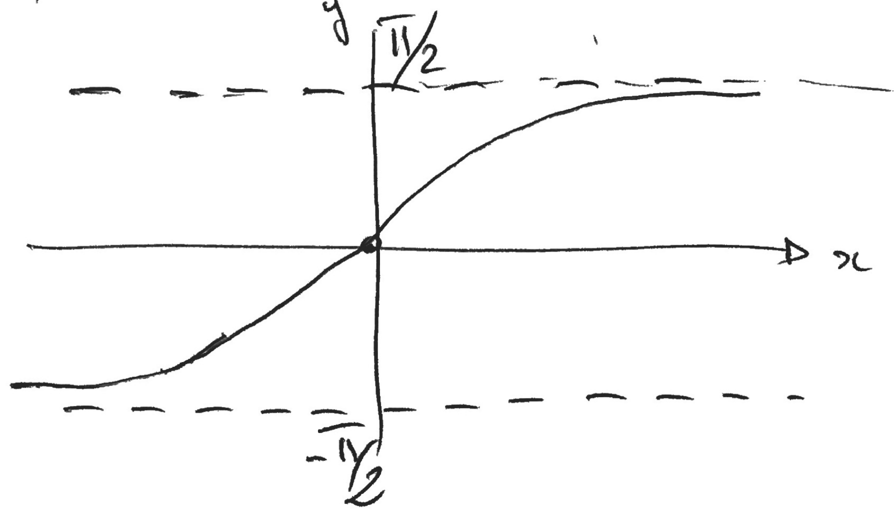
* Inverse of tangent

$$y = \tan^{-1} x \Rightarrow \tan y = x \qquad -\frac{\pi}{2} < y < \frac{\pi}{2} .$$

Example: Evaluate $\tan^{-1}(1)$.

We are asking $\tan y = 1$.

From the unit circle, we have $y = \frac{\pi}{4}$.



$$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2} \qquad \lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

$$\tan(\tan^{-1} x) = x \qquad \tan^{-1}(\tan x) = x.$$

$$f(x) = \tan x \qquad g(x) = \tan^{-1} x.$$

$$g'(x) = \frac{1}{f'[g(x)]} = \frac{1}{\sec^2(\tan^{-1} x)}$$

We know that $y = \tan^{-1} x \Rightarrow \tan y = x$.

The denominator is:

$$\sec^2(\tan^{-1} x) = \sec^2 y.$$

$$\cos^2 y + \sin^2 y = 1$$

$$1 + \tan^2 y = \sec^2 y.$$

$$\Rightarrow \sec^2(\tan^{-1} x) = \sec^2 y = 1 + \underbrace{\tan^2 y}_{x} = 1 + x^2$$

$$\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}}$$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}.$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}.$$

Example: Differentiate the following functions.

a) $f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$.

$$\Rightarrow f'(t) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}.$$

b) $y = \sqrt{z}\,\sin^{-1}(z)$.

$$y = z^{+1/2} \cdot \sin^{-1}(z)$$

$$y' = \frac{1}{2} z^{-1/2} \cdot \sin^{-1}(z) + \frac{\sqrt{z}}{\sqrt{1-z^2}}.$$