# Graphs Galore!

Algebraic and Graphical Translation of Polynomials

A TELE for Desmos and Conceptual Representation

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**The teacher version of the artifact can be accessed at https://teacher.desmos.com/activitybuilder/custom/58e6e9fa3e588a060f483868 and the student version can be accessed at https://student.desmos.com/?prepopulateCode=zst5x using class code ZST5X.**

## Problem Area and Academic Background

As a middle years and secondary teacher, I have had the opportunity to work with students at a range of levels in mathematics between grade 5 and grade 12. In this time, I have observed secondary school students experience difficulty understanding what the parameters of polynomial equations represent and how they affect a function. For example, they do not naturally understand that the *b* in $y=mx+b$ is not the same as the *b* in $y=ax^2+bx+c$. This TELE is intended to support students in developing a deeper understanding of the construction and meaning of polynomials from the linear level to the cubic level. Students at different levels may have different exit points from the sequence. "The National Council of Teachers of Mathematics (NCTM) identified the ability to translate among mathematical representations as a critical skill for learning and doing mathematics" (Adu-Gyamfi, Bossé, & Swift, 2012, p. 159). As linear and other polynomial relationships form the backbone of much of secondary mathematics curricula, it is important that students have a strong understanding of these concepts in order to be able to move forward in their mathematics learning.

The difficulty students experience in connecting abstract algebraic representations and concrete graphical representations is situated within academic literature. According to Renate Nitsch et al (2014), "Despite there being different emphases within the studies themselves, a general agreement exists that, using different basic mathematical forms of representations and translating between these forms, are considered key skills in mathematics" (p.658). This central core importance of student understanding of connections between different forms points to the importance of addressing this challenge. As a result of their study of 645 German ninth and tenth grade students, Nitsch et al (2014) concluded that “For students to develop a holistic understanding of the concept of mathematical functions, they have to be able to identify the connecting elements of a functional dependency and to combine these” (p. 673). Also significant for the purposes of this TELE, the authors refer to a study by Michael O.J. Thomas, Anna J. Wilson, Michael C. Corballis, Vanessa K. Lim, and Caroline Yoon entitled “Evidence from cognitive neuroscience for the role of graphical and algebraic representations in understanding function.”

In their study of student brain activity and strategy use while working with different forms of mathematical representation, Thomas et al found that “experts focused more on the essential characteristics of a function, which helped them execute the translation. In contrast, novices tried to capture the representation as a whole without identifying the key properties relevant to the translation” (Nitsche et al, 2014, p. 658). This finding indicates that understanding of the actual components of an algebraic equation is a factor in the effectiveness and ease of operations performed on and with that function. Students therefore need to understand the role of individual components of equations in order to be able to truly understand the nature of the function as whole. The authors of the study support the need to address incomplete or inaccurate understandings of translating between forms, as “Function deserves attention since it is one of the fundamental concepts of high school and university mathematics, and yet it is often misunderstood by students and teachers” (Thomas et al, 2010, p.607). Misconceptions or incomplete understandings of such functions therefore need to be addressed and built upon so that students may move into higher level mathematics.

Amy L. Nebesniak, an assistant professor, and A. Aaron Burgoa, an eighth-grade teacher, use their own classroom experiences to describe the difference between students working with quadratic equations in vertex form using a set of memorized ‘magic’ rules and students arriving at an understanding of quadratics in vertex form using a more conceptual approach. They describe how “For a number of years, we provided students with the vertex formula, and they successfully graphed by substituting values into the formula. Yet when asked where the formula came from or how it connected to the defining characteristics of quadratics, students did not know. They were performing procedures using this “magical” formula but did not understand how the formula developed” (Nesbesniak & Burgoa, 2015, p.429). Similar to the experiences of Heather and other students in the documentary *A Private Universe*, using the original approach it was possible for students to demonstrate a surface understanding of a concept, but not actually have an accurate or complete deeper understanding. It is therefore important to guide students in learning not simply the mechanics of calculation, but rather, the processes of doing mathematics to build understanding.

The academic research supports the need for mathematical learning experiences that are engaging, immersive, and active. Rote memorization does not translate into meaningful learning on its own. Guided learning experiences that allow students to work with the concepts themselves and not just the rules provide more dynamic learning experiences that promote deeper understandings. The manipulation and observation of change supports students in identifying and addressing their misconceptions, and the process provides valuable assessment data for educators. Students with only partial, yet accurate, understandings also have the opportunity to use this knowledge to further develop their holistic understanding of the relationships. Working through the T-GEM process engages students in meaningful learning experiences that simulate processes used by professional mathematicians.

**Design of a Learning Experience**

The TELE's design follows the Technology-enhanced Generate-Evaluate-Modify (T-GEM) model, in which students build, test, and assess hypotheses regarding relationships in a cyclical fashion to build an understanding. This approach facilitates the development of student understanding of the connections between elements, not simply the application of steps and formulae. The initial foundation of the process is the students' own predictions or educated guesses as to the relationships. As students work collaboratively with peers, they are also required to justify their own perspectives, a process that requires a deeper level of reasoning. As students move through the steps of creating, testing, and revisiting hypotheses regarding the role of parameters within polynomial functions, starting with linear equations and moving on to quadratics and cubics, an increasing level of confounding circumstances are introduced. In her description of the work of Imre Lakatos, Magdalene Lampert (1990) explains Lakatos's perspective that "mathematics develops as a process of "conscious guessing" about relationships among quantities and shapes, with proof following a "zig-zag" path starting from conjectures and moving to the examination of premises through the use of counterexamples or refutations" (p.30). This description supports GEM pedagogy. The sequential approach enables students to progressively build on prior knowledge and integrate new learning in chunks, rather than risking student dismissal of larger and seemingly less connected challenges to previous understanding. Students actually observe the impact their parameter changes have on the graphs of functions. They make the decisions about how best to test or confirm their ideas. The collective set of active experiences allows students to conceptually understand the relationship between algebraic equations and graphical representations of polynomial functions, rather than simply memorizing rules without understanding why or how they are derived or what they mean. Building a conceptual understanding better enables students to engage in higher-level problem solving.

Samia Khan (2012) explains that a shortfall of unstructured use of online applications is “they have limited capacity to guide students, prompt questions, or promote problem solving. This contributes to poor uptake in science classrooms and “clicking without thinking” among students” (p. 59). This same observation can be applied to mathematics classrooms, where students without guided structure often do not actually engage in the intended learning experiences. In this TELE, students are provided with a structure to guide them through their explorations and discussions. The teacher provides both the framework for the learning and how that structure will be used in the classroom environment. While students are leading their learning, the teacher takes on the role of facilitator, guide, and director. “With active guidance from the teacher, these freely available web applications provide a unique environment for students to collaborate with their peers to create, disseminate, test, and refine their scientific ideas” (Khan, 2012, p. 62). When the teacher can guide the students and the students can determine how their path is followed, experiential learning that maximizes time and resources can occur.

When describing previous research in the area of mathematical psychology, Gerald A. Goldin (1998) explains that "There developed a consensus that powerful problem solvers employ powerful heuristic methods, but the techniques proved difficult to teach directly…Student belief systems were identified as important, powerful facilitators of problem-solving success, or else obstacles to it" (p. 138). Goldin's perspective indicates the importance of identifying student belief systems in order for both students and teachers to move forward in mathematics. Accurate and complete belief systems can support students who engage in complex problem solving involving polynomial representations, while belief systems based on misconceptions form opposition to this growth and expansion of learning. Using a T-GEM approach to the concepts and relationships involved in polynomials provides students with opportunities to further develop problem solving skills, reflect on their belief systems, and engage with others in discussions to reconcile their beliefs and their evidence. Because T-GEM pedagogy involves learners discovering learning for themselves, it can be described as an example of a heuristic method.

**Pedagogical Goals of TELE**

The goals of this TELE are as follows:

- Students to develop understanding of the connections between the parameters of polynomial equations and graphical representations.
- Students to discuss mathematical concepts of polynomials using related vocabulary when communicating with peers and teachers.
- Release of explicit teacher control in the learning experience as students are given license to develop and test their own hypotheses with minimal intervention from the teacher in the process.

The overarching goal is for students to develop a meaningful conceptual understanding of polynomials in a way that allows them to have a rich awareness of connections and significance. Using an approach similar to the scientific method facilitates this development as students drive their explorations and work together to actively generate understanding and reflect on their thinking and learning. This understanding can be directly derived in part from the social discussions between students. As students learn how to talk about math, they develop a better understanding of the ideas they are trying to explain. Additionally, as students have opportunities to listen to peers, they are introduced to vocabulary and phrasing that may be novel to them. The risk associated with this process can create the conditions necessary for growth and change in thinking, as “it requires the admission that one's assumptions are open to revision, that one's insights may have been limited, that one's conclusions may have been inappropriate. Although possibly garnering recognition for inventiveness, letting other interested persons in on one's conjectures increases personal vulnerability (Lampert, 1990, p.31).

"To challenge conventional assumptions about what it means to know mathematics, then, teachers and students need to do different sorts of activities together, with different kinds of roles and responsibilities" (Lampert, 1990, p. 35). This TELE can be used as either a self-paced task or a whole class guided task. In either scenario, students are active participants in their learning. Students working at their own pace will move through the phases based on their readiness for each level, building self-reflection and self-regulatory skills. As pairs move at different paces, they may end up regrouping with different pairs at each phase of discussion, which can further increase the potential value of the discussions. With more perspectives, students have more opportunities to hear other rationales and to explain their own thinking. The more varied the discussions, the more information students have with which to work. As a class-paced learning experience, the teacher takes a more active role in determining when and how students move into a new activity. This can be particularly beneficial for students who are still in the early stages of developing self-regulatory skills. Ideally, the teacher would progressively transfer more of the responsibility to the students, either throughout this TELE or the next time students engage in a similar set of activities.

The TELE is applicable across a range of student abilities with the potential for multiple entry and exit points. Students can begin with linear equations and only complete that component, or can move through quadratics and end their activity at that point, or move through to cubics. Students can also begin their explorations at the quadratic level if they already have a solid understanding of linear relationships.

Additionally, higher order polynomials such as quartics and quintics could be added as subsequent phases for students needing additional challenge or in more advanced courses.

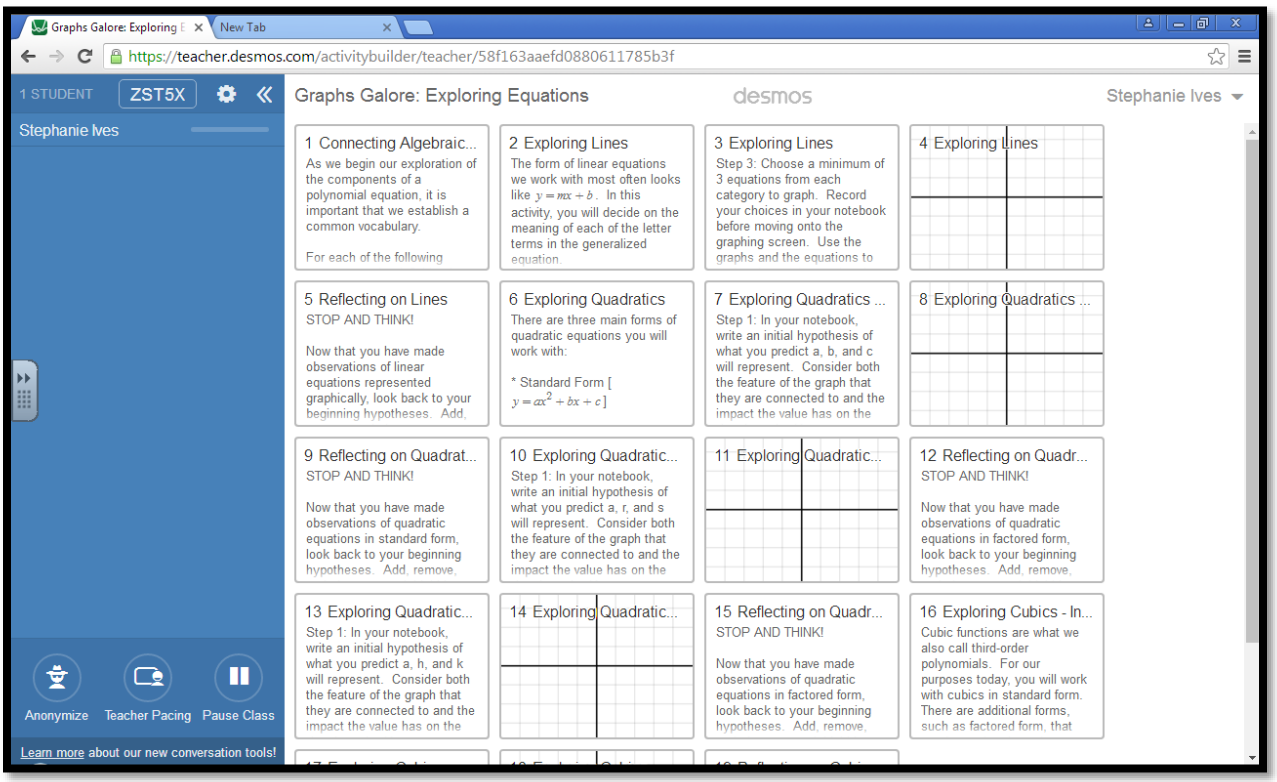
## Digital Technology

Graphing by hand, while also a valuable skill, can be a very tedious and frustrating process for students, especially when many graphs are required. Additionally, many polynomials can be essentially impossible for students to graph accurately by hand. As the focus of this learning experience is on the equations and functions themselves rather than on the graphing process, technology will be used to support this process and enable students to effectively compare results efficiently and accurately. The main technological component of this TELE is Desmos. Desmos is a digital graphing program available both online and in app form, free of charge. Because it is free to use, It can be used as an equalizing measure in terms of socioeconomic status. It can accessed on any Internet enabled device. Jon Orr, a mathematics curriculum leader and teacher in Chatham, Ontario, explains that “In my classroom, Desmos calculator has been a game-changer for student understanding of relationships between graphs and algebraic representations of functions” (Orr, 2017, p.549).

When compared with a traditional Texas Instruments graphing calculator such as the TI-83 used in many classrooms, Desmos offers a more streamlined interface, particularly when used on a touchscreen device. The entry fields are easy to recognize and find, rather than being buried in submenus, and the window settings can be toggled with simple finger strokes on the screen surface, or with the clickable zoom option on the side of the window. Students are able to plot multiple graphs concurrently, and Desmos will automatically colour code the entries for easier comparison. Points on the graphs are clickable for coordinates, and significant points such as intercepts or points of intersection become denoted with a dot for easy recognition. A feature particularly well suited for the purposes of this TELE is the capacity to create sliders for parameters rather than explicitly defining each parameter each time as part of a new equation. Sliders enable a student to adjust a specific parameter easily and observe the effect or transformation in progress. Desmos is therefore largely aligned with the ways in which students typically use technology tools, facilitating use. When choosing a device for access, a device with a larger screen is preferable for ease of use and broad display; however, even a smaller screen can provide access. The ease of use is echoed by David Ebert, a secondary mathematics teacher from Oregon, who writes that "Although other types of software allow students to do everything mentioned in this article, Desmos is an easy-to-use, intuitive, powerful tool that should be explored by any mathematics teacher who teaches the graphing of equations" (Ebert, 2015, p.390).

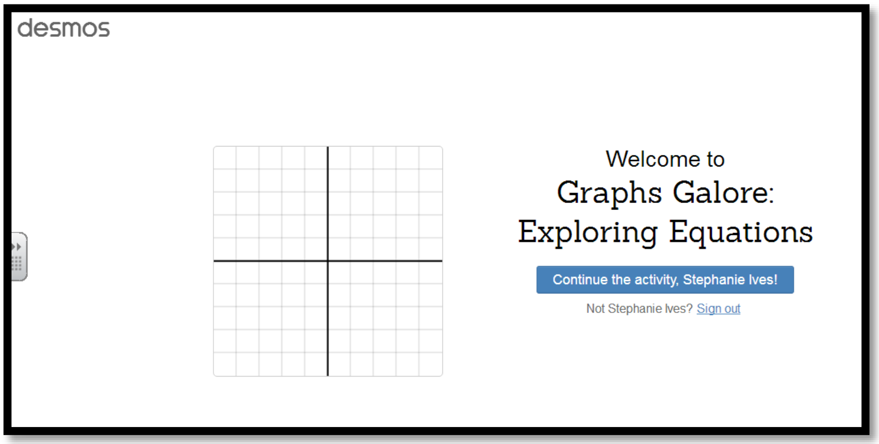
The Desmos online platform includes a teacher component that enables teachers to access learning activities designed by other teachers, as well as to create learning activities of their own. The artifact for this TELE is designed within the Desmos framework. While Desmos does not have many of the aesthetic features available elsewhere on the Internet, it offers all of the components required for a hands-on learning experience that can be either student- or teacher-guided. Without all of the 'bells and whistles' students can focus on the actual activities and content with less distractibility. When used on a iPad, teachers also have the ability to use the guided access feature to keep students on the Desmos app.

Within the teacher dashboard on an active class, the teacher has options available to control the pacing of students moving through the slides, pause the class by temporarily preventing interaction with the slides, and assign randomized code names to the students to anonymize their work. It is also possible for the teacher to view overlays of student graphs and combine into screenshots as desired. Such features enable the teacher to better support students throughout the learning environment.



## Artifact

**The teacher version of the artifact can be accessed at https://teacher.desmos.com/activitybuilder/custom/58e6e9fa3e588a060f483868 and the student version can be accessed at https://student.desmos.com/?prepopulateCode=zst5x using class code ZST5X.**




The TELE artifact is a series of learning activities constructed within the Desmos community. The use of the sequence can be approached from multiple perspectives depending on the needs of a particular group. As largely addressed in previous sections of this paper, adjustments can be made in the areas of delivery method, pacing, teacher involvement, entry and exit points, and assessment.

There are multiple options for delivery method. A student participating in an independent study course can engage in the TELE without the face-to-face discussions. In an online course, discussion could be facilitated using a secondary online tool such as discussion boards, forums, Padlet, OneDrive, or a Google Doc. In class, students can interact in face-to-face pairs and groupings for discussion. Groupings can be either fixed or flexible depending on the class dynamics and teacher knowledge of the students. Delivery method will be closely related to the pacing options that are possible. Depending on teacher preference, there are options for self-paced and group-paced work. A group-paced setup would likely involve exploring linear relations in one class, quadratics in a second, and cubics in a third, with each class building upon the prior. Due to the cyclical nature of the T-GEM process, it is likely there will be some fluidity of movement between the different types of polynomials as students examine the nature and origin of their assumptions.

Assessment directly connected to the activity sequence is formative in nature. The teacher associated with the Desmos teacher account can use a class code to collect data on student participation and interaction within the program. Anecdotal observations of student discussions can also provide additional qualitative assessment data to direct future instruction. Potential associated assessment activities may include reflective journals, challenge problems, and student interviews. The concepts explored in this TELE could also be later expanded into a summative assessment at the end of a unit. One such possibility could be designing a visual image by creating and graphing a series of equations in various forms. Summative assessment would depend largely on the other activities and components of the larger unit of study.

This TELE is envisioned as situated before direct teacher instruction on the parameters of equations. If the teacher introduces the parameters prior to the activity, students will be striving to prove a hypothesis made by the teacher rather than constructing their own predictions. The social element of the discussion checkpoints reinforces the idea that the students are the drivers of the exploration, not the teacher. While the teacher is still present as a facilitator and may introduce additional confounding factors and questions to challenge student thinking, the role of the teacher in this TELE is not to validate student findings. That responsibility remains with the students themselves.

The Desmos activity created for this TELE is available publically within the Desmos community. This allows other educators to access the existing file, as well as to make a copy to their own account and customize it as desired without affecting the original file.

**Bibliography**

Adu-Gyamfi, K., Stiff, L. V., & Bossé, M. J. (2012). Lost in translation: Examining translation errors associated with mathematical representations. *School Science and Mathematics,112*(3), 159-170. doi:10.1111/j.1949-8594.2011.00129.x

Ebert, D. (2015). Graphing projects with Desmos. *The Mathematics Teacher,108*(5), 388-391. doi:10.5951/mathteacher.108.5.0388

Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *The Journal of Mathematical Behavior,17*(2), 137-165. doi:10.1016/s0364-0213(99)80056-1

Khan, S. (2012). A hidden GEM: A pedagogical approach to using technology to teach global warming. *The Science Teacher*, 79(8).

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal,27*(1), 29-63. doi:10.3102/00028312027001029

Nebesniak, A. L., & Burgoa, A. A. (2015). Developing the vertex formula *mean*ingfully. *The Mathematics Teacher,108*(6), 429-433. doi:10.5951/mathteacher.108.6.0429

Nitsch, R., Fredebohm, A., Bruder, R., Kelava, A., Naccarella, D., Leuders, T., & Wirtz, M. (2014). Students' competencies in working with functions in secondary mathematics education—Empirical examination of a competence structure model. *International Journal of Science and Mathematics Education,13*(3), 657-682. doi:10.1007/s10763-013-9496-7

Orr, J. (2017). Function transformations and the Desmos Activity Builder. *The Mathematics Teacher,110*(7), 549-551. doi:10.5951/mathteacher.110.7.0549

Thomas, M. O., Wilson, A. J., Corballis, M. C., Lim, V. K., & Yoon, C. (2010). Evidence from cognitive neuroscience for the role of graphical and algebraic representations in understanding function. *ZDM,42*(6), 607-619. doi:10.1007/s11858-010-0272-7

## Appendix A: Teacher Prompts

As your students work through the TELE, there will be opportunities for you to ask questions that encourage them to consider other circumstances or reflect on gaps in their decisions without giving them any information. These questions are intended to support students in their reflections, not as a means of you as the teacher directly evaluating or commenting on their thinking. Alternately, these questions could form prompts for math journal entries. A bank of potential questions is included below.

Linear Relations

- What does it mean when there is no *b* value?
- How is the graph different when there is only one variable present as compared to when there are two?
- What other representations are possible?
- If an algebraic expression is written in a form other than y=mx+b, do the same rules apply?

Quadratic Relations

- How are your observations of quadratic relationships similar to or different from your observations of linear relationships?
- When do you think each of the forms would be most useful?
- In factored form, how does the sign of r and s connect to the graph?
- In factored form, what happens when there is a coefficient other than 1 with the x?
- In vertex form, how does the sign of h and k connect to the graph?
- How is the graph affected when a value is “missing”?

Cubic Relations

- How do the end behaviours relate to those of linear and quadratic relations?
- How is the graph affected when a value is “missing”?
- How do you think the parameters would translate into factored form?
- How are the parameters related to the parameters of functions of other degrees?

**Appendix B: Note Record Sheets**

Name:______________________________ Partner:__________________________________
Type of Polynomial:_____________________________________________

Initial Hypotheses:

Notes from Partner Discussion:

Observations:

| **Equation** | **Observations** |
|---|---|
| | |

Updated Hypotheses:

Notes from Group Discussion:

Final Conclusions: