Some Matrix Theory

With this brief theoretical interlude we want to dive deeper into analysing the structure of the discretised POISSON-DIRICHLET (and other elliptic) problems. The better we understand the properties of the 'big linear system' $L^h u^h = f^h$, the better we will understand what characteristic features of the continuous problem $Lu = f$ are preserved under a 'suitable' discretisation scheme. Furthermore, we will later use all the information that we can possibly gather on the matrix L^h to select a numerical method for solving the discrete problem that is guaranteed to converge, and that additionally exploits all the structure of L^h to compute u^h as efficiently as somehow possible.

2.1.7 Definition (Diagonal Dominance) If in the i^{th} row of a matrix $A \in \mathbb{R}^{n \times n}$ the absolute value of the diagonal entry is

- greater than or equal to the sum of the absolute values of the off-diagonal terms

  then we say that A is *weakly diagonally dominant* in this row;
- greater than the sum of the absolute values of the off-diagonal terms

  then we say that A is *strictly diagonally dominant* in this row.

A matrix A with the properties that

(i) A is weakly diagonally dominant in all rows

(ii) A is strictly diagonally dominant in at least one row

(iii) for all rows i_0 there exists a chain of indices $i_0 \to i_1 \to \cdots \to i_s$ to a strictly diagonally dominant row i_s such that all $a_{i_{l-1} i_l} \neq 0$ $(l = 1, \ldots, s)$

is called *weakly chained diagonally dominant*. If, instead of (iii), there even holds that

(iv) for any two rows i_0, i_s there exists a chain of indices $i_0 \to i_1 \to \cdots \to i_s$ such that all $a_{i_{l-1} i_l} \neq 0$ and all $a_{i_l i_{l-1}} \neq 0$ $(l = 1, \ldots, s)$

then A is called *irreducibly diagonally dominant*.

The two chain properties describe how data from the right hand side and information from each component of the solution propagate through the linear system. As can be seen from the conditions (iii) and (iv), they describe the structure of the sparsity pattern of A.

With the weaker chain property (iii), information from row i_s is referred to in row i_{s-1}. Then the equation in row i_{s-2} refers to the component i_{s-1} of the solution, and hence indirectly also to i_s. Finally, row i_0 directly or indirectly depends on the components $i_1, i_2, \ldots, i_s$ of the solution and the right hand side. It is not required for (iii) that conversely row i_s also depends on row i_0.

This is the difference to the stronger property (iv). Here, information is shared globally and all rows directly or indirectly refer to themselves and all other rows.

2.1.8 Definition (Monotone Matrix) A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *(inverse-)monotone* if for all vectors $u \in \mathbb{R}^n$

It is a simple exercise to show that a matrix A is monotone if and only if A^{-1} exists and all its entries are positive or zero: $(A^{-1})_{ij} \geq 0, \forall i, j = 1, \ldots, n$.

2.1.9 Definition (Z-, L- and M-Matrices) A matrix $A \in \mathbb{R}^{n\times n}$ is called

- *Z-matrix* or *L_0-matrix* if all off-diagonal entries are negative or zero:

- *L-matrix* if all off-diagonal entries are negative or zero and all diagonal entries are positive:

- *M-matrix*, if it is a monotone Z-matrix.

There are many characterisations of M-matrices. The following sufficient condition is the most illustrative one in the context of discretised elliptic PDEs:

2.1.10 Lemma (M-Criterion) *If $A \in \mathbb{R}^{n\times n}$ is a weakly chained diagonally dominant L-matrix, then A is also monotone and hence an M-matrix.*

Stability of Finite Difference Methods

All we have to do now is to verify that the matrix L^h of the discretised POISSON equation (2.4) is an M-matrix. Then we have already answered the existence-and-uniqueness question in the definition of stability.

2.1.11 Lemma (Discrete Laplacian is an M-Matrix) *The discretised elliptic operator L^h in the* POISSON-DIRICHLET *problem (2.4) is an M-matrix, independent of h.*

Proof. The matrix L^h is

- strongly diagonally dominant in all rows

- weakly, but not strongly diagonally dominant in all rows

Obviously, the matrix L^h is irreducibly diagonally dominant[*], and in particular it is weakly chained diagonally dominant[†]. It is also obvious that all off-diagonal entries are non-negative while all diagonal entries are positive and hence that L^h is an L-matrix. The M-criterion now implies that L^h is an M-matrix, as asserted. □

The (inverse) monotonicity of L^h admits a very desirable conclusion beyond existence and uniqueness of discrete solutions u^h:

2.1.12 Theorem (Discrete Elliptic Maximum Principle) *Let the discretised elliptic operator $L^h \in \mathbb{R}^{n\times n}$ be a weakly chained diagonally dominant L-matrix. Then*

$$L^h u^h \le 0 \qquad \text{in } \Omega^h \qquad\qquad \Rightarrow \qquad\qquad \max_{x\in\bar\Omega^h} u^h(x) \le \max_{x\in\partial\Omega^h} u^h(x),$$

i.e. the discrete solution u^h assumes its maximum on the boundary.

Furthermore, the strong connectivity (irreducibility) of the system matrix L^h reveals that even a *local* change in only one component of the right hand side f^h immediately affects the entire solution u^h *globally*. This phenomenon of infinitely fast propagation of information through the entire domain is characteristic for elliptic equations.

2.1.13 Theorem (Stability Inequality) *For the discrete* Poisson-Dirichlet *problem (2.4) we have the stability inequality*

$$\max_{\bar\Omega^h}|v^h| \le \frac{(\operatorname{diam}\Omega)^2}{4}\max_{\Omega^h}|L^h v^h| + \max_{\partial\Omega^h}|v^h| \tag{2.9}$$

for all grid functions v^h.

Proof. This proof can be interpreted as a ‘discretisation’ of the corresponding proof for the continuous boundary value problem, which uses Green’s functions.

---

[*] One can walk from any interior grid point to any other interior grid point, taking only steps that are covered by the 5-point stencil.

[†] One can walk from any interior grid point to a point adjacent to the boundary (where the matrix is strictly diagonally dominant).

For each grid point $y \in \bar{\Omega}^h$, we define a corresponding discrete GREEN's function G_y^h as the solution of the problem

$$L^h G_y^h(x) = \frac{1}{h_1 h_2}\delta_{x,y} \qquad \text{in } \Omega^h \qquad\qquad G_y^h(x) = \delta_{x,y} \qquad \text{on } \partial\Omega^h. \tag{2.10}$$

For any two points $x, y \in \bar{\Omega}^h$, the KRONECKER-delta δ_{xy} returns 1 if $x = y$, 0 otherwise. Therefore, all data are nonnegative and the discrete maximum principle implies that $G_y^h \geq 0$ as well.

By means of these discrete GREEN's functions, one can express any grid function v^h as

$$v^h(x) = h_1 h_2 \sum_{y \in \Omega^h} G_y^h(x) L^h v^h(y) + \sum_{y \in \partial\Omega^h} G_y^h(x) v^h(y) \tag{2.11}$$

with $x \in \bar{\Omega}^h$, since both the left hand side and the right hand side of this expression solve the problem

$$L^h u^h = v^h \qquad \text{in } \Omega^h \qquad\qquad u^h = v^h \qquad \text{on } \partial\Omega^h. \tag{2.12}$$

Due to the invertibility of L^h, this discrete boundary value problem admits a unique solution and hence the two sides of (2.11) are indeed equal.

For any point $x \in \bar{\Omega}^h$, (2.11) yields the estimate

$$|v^h(x)| \leq h_1 h_2 \sum_{y \in \Omega^h} G_y^h(x) \max_{\Omega^h} |L^h v^h| + \sum_{y \in \partial\Omega^h} G_y^h(x) \max_{\partial\Omega^h} |v^h(y)|. \tag{2.13}$$

To further estimate the first sum in (2.13), we consider the quadratic function $q(x) = \frac{1}{4}|x - y|^2$ that is centred around a grid point $y \in \bar{\Omega}^h$. Since all third derivatives of q vanish, the central differencing scheme is exact for this function, i.e. $L^h q|_{\Omega^h} = -\Delta q = -1$ with a truncation error of zero. For the sum in the first term of (2.13) we introduce the abbreviation

$$w^h = h_1 h_2 \sum_{y \in \Omega^h} G_y^h(x)$$

for which we have

$$L^h w^h = 1 \qquad \text{in } \Omega^h \qquad\qquad w^h = 0 \qquad \text{on } \partial\Omega^h$$

and hence, since $|x - y| \leq \operatorname{diam} \Omega$,

$$L^h(q|_{\Omega^h} + w^h) = 1 \qquad \text{in } \Omega^h \qquad\qquad q|_{\Omega^h} + w^h \leq \frac{(\operatorname{diam} \Omega)^2}{4} \qquad \text{on } \partial\Omega^h.$$

From the discrete maximum principle we infer

$$\max_{\bar{\Omega}^h} h_1 h_2 \sum_{y\in\Omega^h} G_y^h(x) = \max_{\bar{\Omega}^h} w^h \leq \max_{\bar{\Omega}^h} q + w^h \leq \max_{\partial\Omega^h} q + w^h \leq \frac{(\operatorname{diam}\Omega)^2}{4}.$$

To estimate the second sum in (2.13), we observe that just like q, the function $c(x) \equiv 1$ is also differentiated exactly: $L^h c|_{\Omega^h} = 0$. This way, (2.11) simplifies to

$$1 = \sum_{y\in\partial\Omega^h} G_y^h(x).$$

After substituting this result back into the inequality (2.13), we obtain the asserted stability estimate. □

Note that if we apply the stability inequality to the numerical solution u^h, we obtain its continuous dependence on the data f^h and g^h, independent of h.

Convergence of Finite Difference Methods

2.1.14 Theorem (Convergence on Equidistant Grids) *Let $\Omega^h \subset \mathbb{R}^2$ be a rectangular grid with constant grid spacing h_1 and h_2 in x_1- and x_2-direction, respectively. Then the finite difference discretisation (2.4) for the* POISSON-DIRICHLET *problem is 2nd order convergent, provided that the solution u to the continuous problem is in $C^4(\bar{\Omega})$. More specifically, the a priori error estimate*

$$\max_{\bar{\Omega}^h} \left|u|_{\Omega^h} - u^h\right| \leq \frac{h^2}{24}(\operatorname{diam}\Omega)^2 \max_{\bar{\Omega}} \left\{\, |\partial^4_{x_1} u|, |\partial^4_{x_2} u| \,\right\} \tag{2.14}$$

holds.

Proof.

□

2.1.15 Theorem (Convergence on Non-Equidistant Grids) *Let $\Omega^h \subset \mathbb{R}^2$ be a Cartesian grid with maximum grid spacing h_1 and h_2 in x_1- and x_2-direction, respectively. Then the finite difference discretisation based on (2.7) for the* POISSON-DIRICHLET *problem is 2nd order convergent, provided* *that the solution* u *to the continuous problem is in* $C^4(\bar{\Omega})$*. More specifically, the a priori error estimate*

$$\max_{\bar{\Omega}^h} \left| u|_{\Omega^h} - u^h \right| \leq \frac{h^2}{24} (\operatorname{diam} \Omega)^2 \max_{\bar{\Omega}} \left\{ |\partial^4_{x_1} u|, |\partial^4_{x_2} u| \right\} + \frac{h^3}{3} \max_{\bar{\Omega}} \left\{ |\partial^3_{x_1} u|, |\partial^3_{x_2} u| \right\} \tag{2.15}$$

holds.

Proof. Due to the reduced consistency order of only 1 in this non-equidistant case, the default approach

$$\text{consistency of order } p \wedge \text{stability} \Rightarrow \text{convergence of order } p$$

would only give linear convergence in h; a result that is not optimal. The quadratic rate of convergence can be derived with a slight refinement of the proof for the stability estimate, applied to the error $e^h = u|_{\Omega^h} - u^h$. □

2.1.16 Remark (C^4-Regularity up to the Boundary) The assumption that the analytical solution belongs to the space $C^4(\bar{\Omega})$ is not normally satisfied, as such a high regularity of the solution usually requires a sufficiently regular domain, with no corners. Even for the problem

$$-\Delta u = 1 \qquad \text{in } \Omega \qquad\qquad u = 0 \qquad \text{on } \partial\Omega$$

with C^∞-data, $u \notin C^4(\bar{\Omega})$ if Ω is, for example, the square $]0,1[^2$: assuming that u is four times continuously differentiable up to the boundary, then the PDE prescribes

$$-\Delta u|_{x=0} =$$

but the boundary condition implies

$$-\Delta u|_{x=0} =$$

Consequently, even with the perfectly smooth data in this example, the corners in the domain do not even allow for second derivatives that are continuous up to the boundary.

2.1.17 Conclusions