Probability meets PDEs: The Numerics of Stochastic Heat Equation

Tim Jaschek

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Let U be an bounded domain in $\mathbb{R}^n$. We will consider a modification of the well known heat equation with homogeneous Dirichlet boundary conditions and given initial condition u_0 and force term f.

$$\begin{cases} \partial_t u(t) - \Delta u = f(t), & \text{in } U \times (0, T) \\ u(0, x) = u_0(x), & \text{in } U \\ u(t) = 0, & \text{on } (0, T) \times \partial U. \end{cases} \tag{1}$$

Examples: Why should we add randomness to model?

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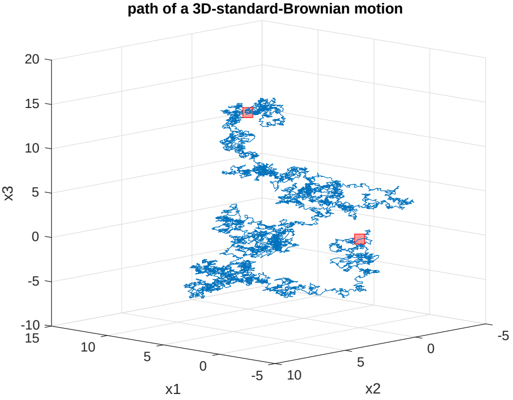
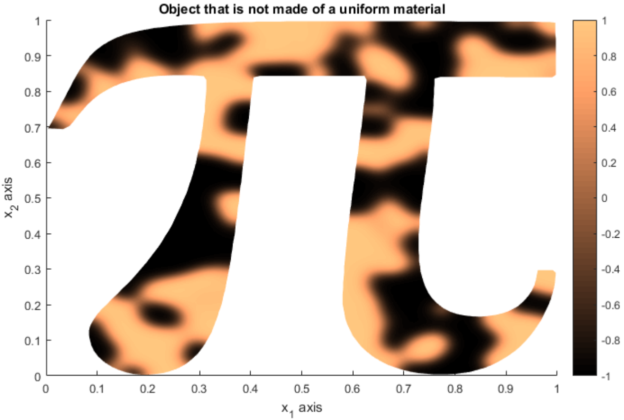
i) Heat is motion of atoms. We can interpret heat transfer as random collisions of particles.



Examples: Why should we add randomness to model?

ii) Heat flow in objects that are not made of a uniform material or that follow a diffusion.



We are interested in the case when the force term is random.

$$\begin{cases} \partial_t u(t) - \Delta u = W(t,x,\omega), & \text{in } U \times (0,T) \\ u(0,x) = u_0(x), & \text{in } U \\ u(t) = 0, & \text{on } (0,T) \times \partial U. \end{cases} \tag{2}$$

Rather than understanding u as a function of time and space we want to see it as a $H = L^2(D)$-valued stochastic process.
We want to write

$$du = \Delta u dt + \sigma dW_t \tag{3}$$

Or in integral form

$$u_t = u_0 + \int_0^t \Delta u_s ds + \int_0^t \sigma dW_s \tag{4}$$

Note that the Integral is a priori not defined since the Q-Wiener process is of infinite variation! $\rightarrow$ generalized Itô Integral

Definition (Notation for Solutions of Stochastic Heat)

A predictable H-valued Process $\{u_t : t \in [0,T]\}$ is called a *strong solution* of the stochastic heat equation if

$$u_t = u_0 + \int_0^t \Delta u_s ds + \int_0^t \sigma dW_s, \quad \mathbf{P} - a.s., \tag{5}$$

weak solution of the stochastic heat equation if for all $v \in \mathcal{D}(\Delta)$:

$$\langle u_t, v\rangle = \langle u_0, v\rangle + \int_0^t -\langle u_s, \Delta v\rangle ds + \int_0^t \langle \sigma dW_s, v\rangle, \quad \mathbf{P} - a.s., \tag{6}$$

and *mild solution* of the stochastic heat equation if

$$u_t = e^{-t\Delta} u_0 + \int_0^t e^{-(t-s)\Delta} \sigma dW_s, \quad \mathbf{P} - a.s.. \tag{7}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E})$ be a measurable space.
$X : \Omega \to E$ is called E-valued random variable if X is $\mathcal{F}$-measurable.
Often we consider cases where $E = \mathbb{R}$ or $E = \mathbb{R}^d$. We will instead consider the case where $E = H$ for a general Hilbert space with the Borel σ algebra $\mathcal{E} = \mathcal{B}(H)$.

Definition ($L^p(\Omega, H)$)

The space $L^p(\Omega, H)$ is the space of H-valued $\mathcal{F}$-measurable random variables with finite p-th moment. It is Banach with the norm

$$\|X\|_{L^p(\Omega,H)} := \left(\int_\Omega \|X(\omega)\|_H^p d\mathbf{P}(\omega) \right)^{\frac{1}{p}} = \mathbf{E}\left[\|X\|_H^p\right]^{\frac{1}{p}}. \quad (8)$$

For $p = 2$ this space is Hilbert with inner product

$$\langle X, Y \rangle_{L^p(\Omega,H)} := \int_\Omega \langle X(\omega), Y(\omega) \rangle_H d\mathbf{P}(\omega) = \mathbf{E}\left[\langle X, Y \rangle_H\right]. \quad (9)$$

Definition (Covariance Operator)

Let H be a Hilbert space. A linear operator $\mathcal{C} : H \to H$ is the covariance of H-valued randomvariables X and Y iff

$$\langle \mathcal{C}\phi, \psi \rangle_H = \mathrm{Cov}\left(\langle X, \phi \rangle_H, \langle Y, \psi \rangle_H\right), \quad \forall \phi, \psi \in H. \tag{10}$$

Definition (H-valued Gaussian random variable)

An H-valued random variable X is called Gaussian iff $\langle X, \phi \rangle_{L^2(\Omega,H)}$ is a real valued Gaussian random variable for all $\phi \in H$.

Theorem

Let X be an H-valued Gaussian with $\mu = \mathbb{E}[X]$. Then $X \in L^2(\Omega, H)$ and the covariance operator $\mathcal{C}$ of X is well-defined trace class operator. We write $X \sim \mathcal{N}(\mu, \mathcal{C})$.

Let $Q \in \mathcal{L}(H, H)$ be non-negative definite, symmetric and such that there exists an orthonormal basis $\{\varphi_i : i \in \mathbb{N}\}$ of eigenfunctions with corresponding eigenvalues $\lambda_i \geq 0$ such that $\sum_{i \in \mathbb{N}} \lambda_i < \infty$.

Definition (Q-Wiener Process)

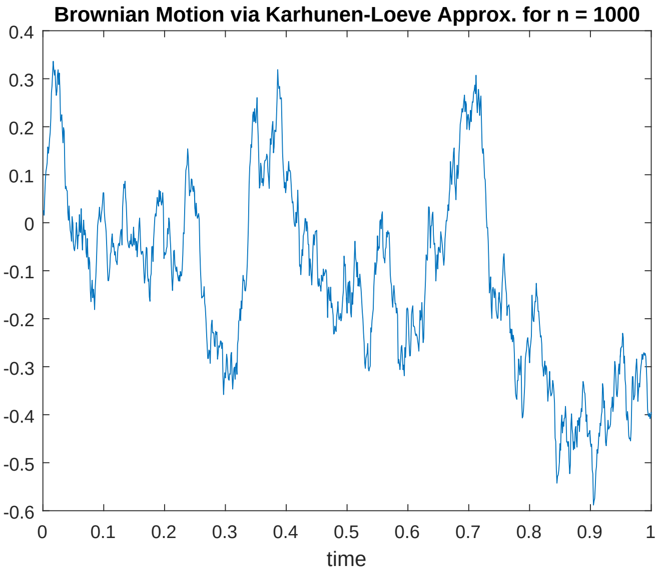
A H-valued stochastic process $\{W_t : t \geq 0\}$ is called Q-Wiener Process if

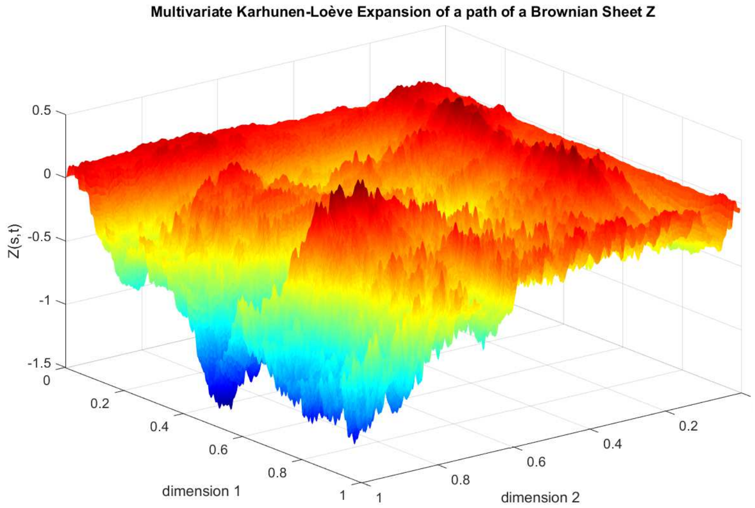
i) $W_0 = 0$ a.s.,

ii) W_t is a continious function $\mathbb{R}_+ \to H$ for each $\omega \in \Omega$,

iii) W_t is $\mathcal{F}_t$-adapted and $W_t - W_s$ is independent of $\mathcal{F}_s$ for $s \leq t$,

iv) $W_t - W_s \sim \mathcal{N}(0, (t-s)Q)$ for all $0 \leq s \leq t$.


Brownian Motion via Karhunen-Loeve Approx. for n = 1000
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time


Multivariate Karhunen-Loève Expansion of a path of a Brownian Sheet Z
0.5
0
-0.5
-1
-1.5
Z(s,t)
0
0.2
0.4
0.6
0.8
1
dimension 1
1
0.8
0.6
0.4
0.2
dimension 2

Theorem (Karhunen-Loève Expansion for Q-Wiener Process)

Let Q statisfy our basic assumptions. Then W_t is a Q-Wiener Process if and only if

$$W_t = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \varphi_i \beta_t^{(i)} \quad a.s. \tag{11}$$

where $\beta^{(i)}$ are i.i.d. $\mathcal{F}_t$-Brownian motions and the series converges in $L^2(\Omega, H)$. Moreover it converges in $L^2(\Omega, \mathcal{C}([0, T], H))$.

We define the stochastic integral with respect to a Q-Wiener Process as

$$\int_0^t X_s dW_s := \sum_{i=1}^{\infty} \sqrt{\lambda_i}\varphi_i \int_0^t X_s d\beta_s^{(i)}, \tag{12}$$

where the integrals with respect to Brownian motion are so called Itô integrals. They are limits in L^2 and not pathwise integrals! It holds

$$\lim_{n\to\infty} \sup_{s\le t} \left| \sum_{k=1}^{p^{(n)}} X_{t_k^{(n)}} \left(B^{(i)}_{t_{k+1}^{(n)}\wedge s} - B^{(i)}_{t_k^{(n)}\wedge s} \right) - \int_0^t X_r dB_r^{(i)} \right| = 0 \tag{13}$$

in probability.

The most interesting case is when $Q = I$. However, our definition of a Q- Wiener process doesn‘t work anymore since I is not trace class.

Definition (space-time white noise)

The cylindrical Wiener process (also called space-time white noise) is the H-valued stochastic process W_t defined by

$$W_t = \sum_{i=1}^{\infty} \varphi_i B_t^{(i)}, \tag{14}$$

in $L^2(\Omega, H)$ where $\{\varphi_i : i > 0\}$ is any orthonormal basis of H and $B^{(i)}$ are iid. Brownian motions.

We take $U = (0, \pi)$. Then $-\Delta$ has eigenfunctions and eigenvalues

$$\varphi_i(x) = \sqrt{2/\pi}\sin(ix), \qquad \lambda_i = i^2.$$

Let now W be a Q-Wiener process be such that Q has the same eigenfunctions as $-\Delta$ with corresponding eigenvalues ξ_i. Then for $v \in \mathcal{D}(\Delta)$ a weak solution statisfies:

$$\begin{aligned}\langle u_t, v\rangle_{L^2(U)} = \langle u_0, v\rangle_{L^2(U)} \quad &+ \quad \int_0^t \langle -u_s, \Delta v\rangle_{L^2(U)} ds \\ &+ \quad \sum_{i=1}^{\infty} \int_0^t \sigma\sqrt{\xi_i}\langle \varphi_i, v\rangle dB_s^{(i)}\end{aligned}$$

Expand $u_t = \sum_{i=1}^{\infty} \hat{u}_t^{(i)}\varphi_i$ for $\hat{u}_t^{(i)} := \langle u_t, \varphi_i\rangle_{L^2(U)}$.

Then for $v \in \mathcal{D}(\Delta)$ a weak solution statisfies:

$$\langle u_t, v\rangle_{L^2(U)} = \langle u_0, v\rangle_{L^2(U)} + \int_0^t \langle -u_s, \Delta v\rangle_{L^2(U)} ds + \sum_{i=1}^{\infty} \int_0^t \sigma\sqrt{\xi_i}\langle \varphi_i, v\rangle dB_s^{(i)}$$

Expand $u_t = \sum_{i=1}^{\infty} \hat{u}_t^{(i)}\varphi_i$ for $\hat{u}_t^{(i)} := \langle u_t, \varphi_i\rangle_{L^2(U)}$. Take $v = \varphi_i$ to see

$$\hat{u}_t^{(i)} = \hat{u}_0^{(i)} - \int_0^t \lambda_i \hat{u}_s^{(i)} ds + \int_0^t \sigma\sqrt{\xi_i} dB_s^{(i)} \tag{15}$$

Hence $\hat{u}^{(i)}$ statisfies the SODE (Ornstein-Uhlenbeck Process)

$$d\hat{u}^{(i)} = -\lambda_i \hat{u}^{(i)} dt + \sigma\sqrt{\xi_i} dB_t^{(i)}. \tag{16}$$

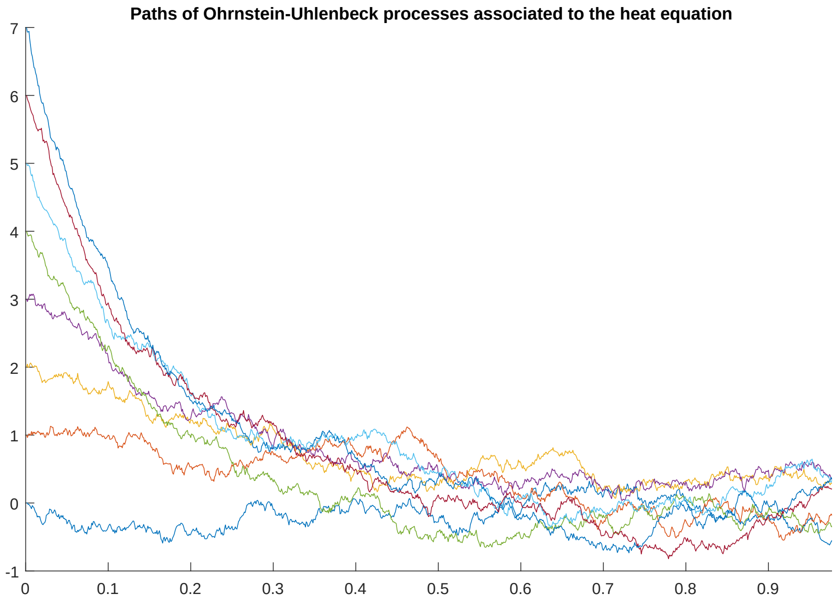
Hence $\hat{u}^{(i)}$ statisfies the SODE (Ornstein-Uhlenbeck Process)

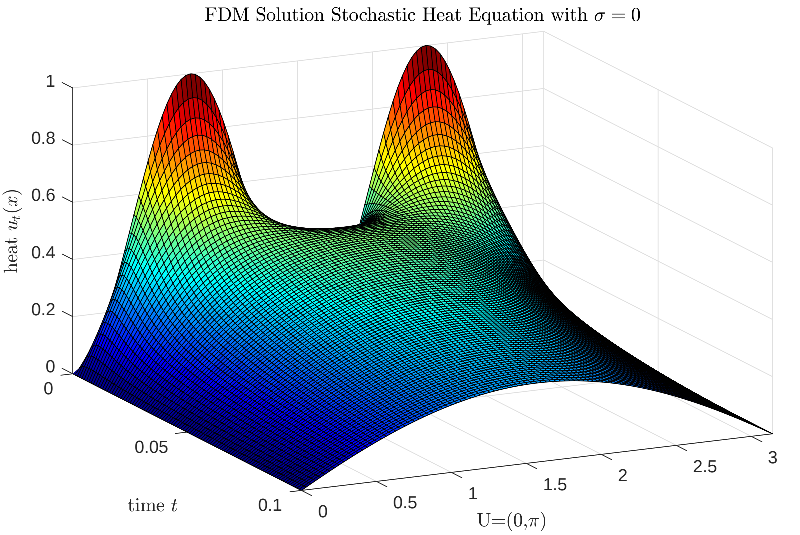
$$d\hat{u}^{(i)} = -\lambda_i \hat{u}^{(i)} dt + \sigma \sqrt{\xi_i} dB_t^{(i)}. \tag{17}$$

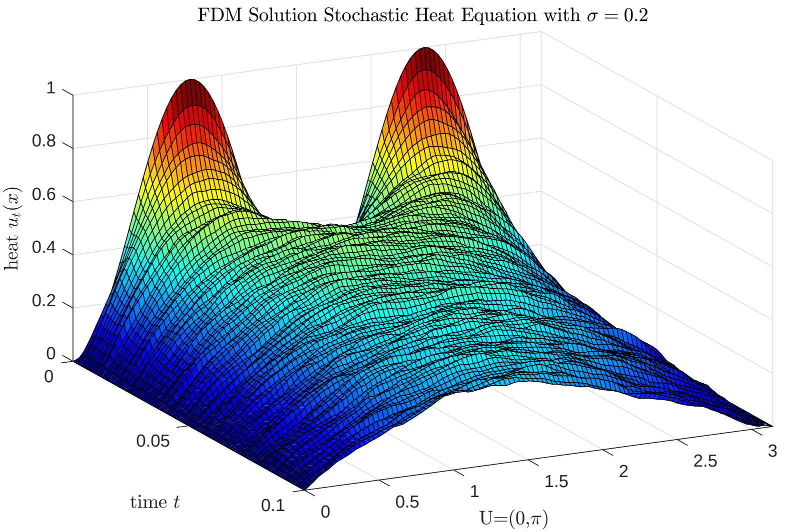
One can show that $\mathrm{Var}(\hat{u}_t^{(i)}) = \frac{\sigma^2 \xi_i}{2\lambda_i}(1 - e^{-2\lambda_i t})$ and thus by Parseval's identity

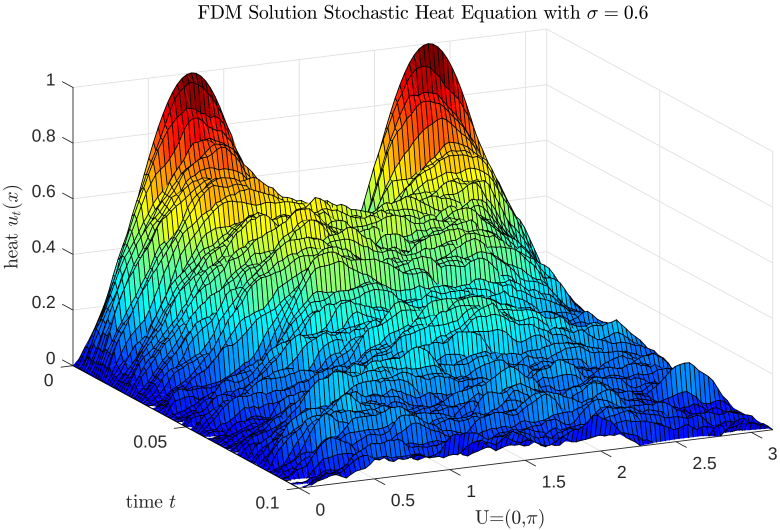
$$\|u_t\|^2_{L^2(\Omega, L^2(0,\pi))} = \mathbb{E}\left[\sum_{i=1}^{\infty} |\hat{u}_t^{(i)}|^2\right] = \sum_{i=1}^{\infty} \frac{\sigma^2 \xi_i}{2\lambda_i}(1 - e^{-2\lambda_i t}) \tag{18}$$

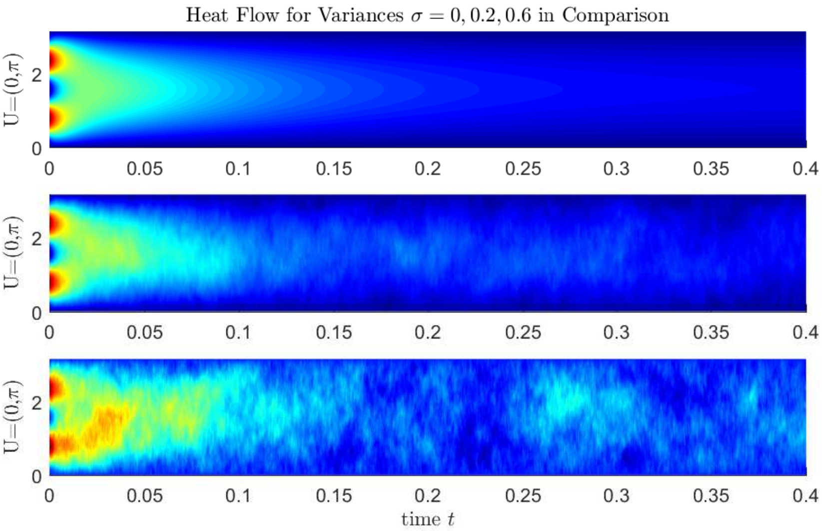
which converges if $\sum_{i=1}^{\infty} \xi_i / \lambda_i$ is finite, which is the case since Q is trace class.


Paths of Ohrnstein-Uhlenbeck processes associated to the heat equation
7
6
5
4
3
2
1
0
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


FDM Solution Stochastic Heat Equation with $\sigma = 0$
heat $u_t(x)$
1
0.8
0.6
0.4
0.2
0
0
0.05
0.1
time t
0
0.5
1
1.5
2
2.5
3
U=(0,π)


FDM Solution Stochastic Heat Equation with $\sigma = 0.2$
heat $u_t(x)$
1
0.8
0.6
0.4
0.2
0
0
0.05
0.1
time t
0
0.5
1
1.5
2
2.5
3
U=(0,π)


FDM Solution Stochastic Heat Equation with $\sigma = 0.6$
heat $u_t(x)$
1
0.8
0.6
0.4
0.2
0
0
0.05
0.1
time t
0
0.5
1
1.5
2
2.5
3
U=(0,π)


Heat Flow for Variances $\sigma = 0, 0.2, 0.6$ in Comparison
U=(0,π)
0
2
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
time t

Now let $U = (0,\pi) \times (0,\pi)$. One can show that $-\Delta$ as eigenvalues $\lambda_{i,j} = i^2 + j^2$. Again let us assume that Q has the same eigenfunctions but with corresponding eigenvalues $\xi_{i,j}$.

$$d\hat{u}^{(i,j)} = -\lambda_{i,j}\hat{u}^{(i,j)}dt + \sigma\sqrt{\xi_{i,j}}dB_t^{(i,j)}. \tag{19}$$

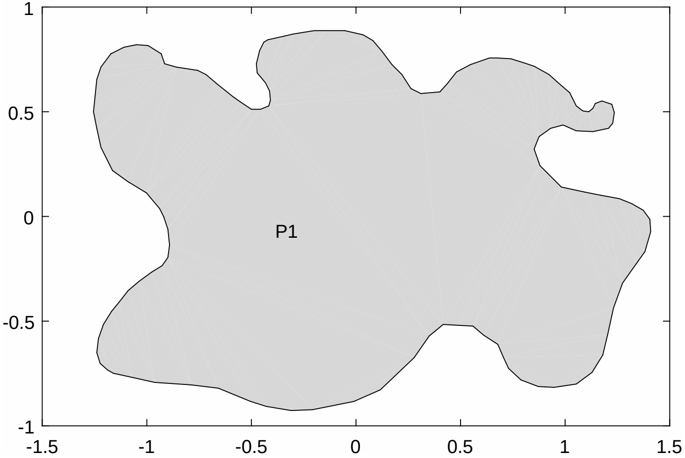
Once again let us apply Parseval's identity to obtain

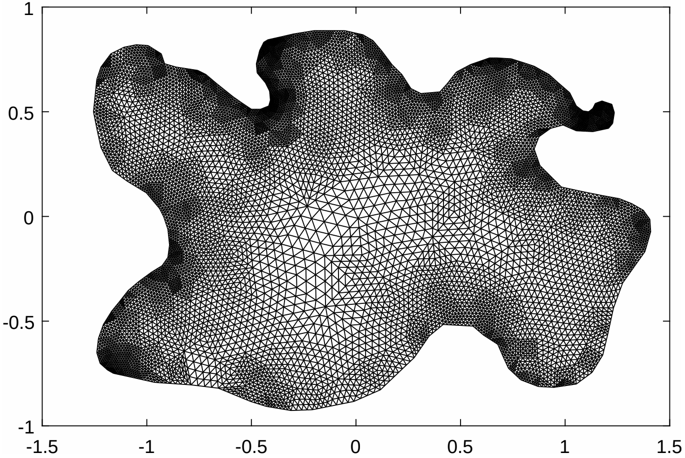
$$\|u_t\|^2_{L^2(\Omega,L^2((0,\pi)\times(0,\pi)))} = \mathbb{E}\left[\sum_{i,j=1}^{\infty} |\hat{u}_t^{(i,j)}|^2\right] = \sum_{i,j=1}^{\infty} \frac{\sigma^2\xi_{i,j}}{2\lambda_{i,j}}(1-e^{-2\lambda_{i,j}t}) \tag{20}$$

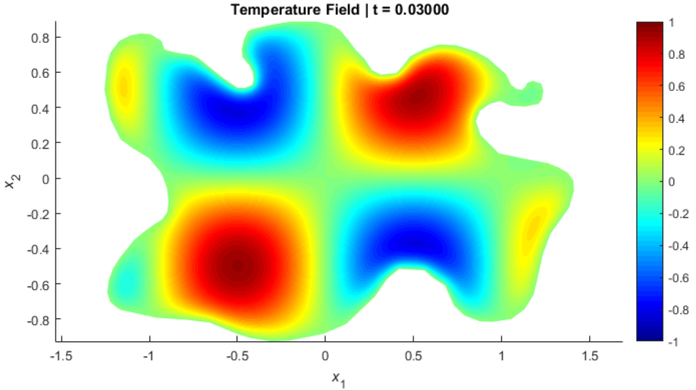
This justifies that u is in $L^2(\Omega, L^2((0,\pi)\times(0,\pi)))$ since Q is trace class. However, for a cylindrical Wiener Process

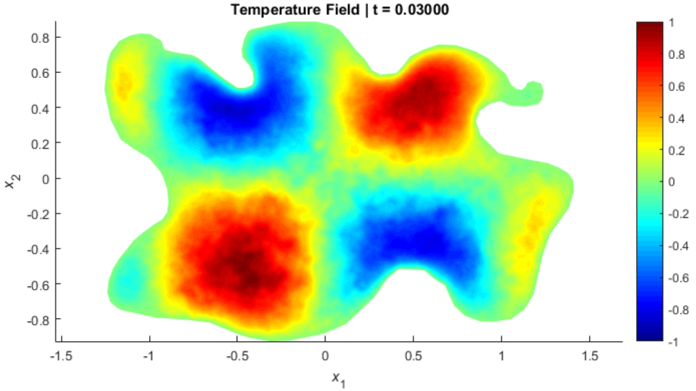
$$\sum_{i,j=1}^{\infty} \frac{1}{\lambda_{i,j}} = \sum_{i,j=1}^{\infty} \frac{1}{i^2+j^2} = \infty, \tag{21}$$

Which implies that there exists no weak solution in this case.


P1
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5


1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5


Temperature Field | t = 0.03000
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
x2
-1.5
-1
-0.5
0
0.5
1
1.5
x1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1


Temperature Field | t = 0.03000
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
x_2
-1.5
-1
-0.5
0
0.5
1
1.5
x_1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

References

- *An Introduction to Computational Stochastic PDEs*, Gabriel J. Lord, Catherine E. Powell, Tony Shardlow, August 2014, Camebridge University Press
- *An Introduction to SPDEs*, Martin Hairer, July 2014, Lecture notes, The University of Warwick

All figures and videos are generated using MATLAB.

Probability meets PDEs: The Numerics of Stochastic Heat Equation

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Abstract

The stochastic heat equation models a heat flow in a material disturbed by a space-time white noise.
To make sense of the stochastic heat equation, the Q-Wiener Process and a generalization of the Itô-Integral with respect to this process will be introduced. Those will lead us to a beautiful relation of weak solutions to the stochastic heat equation in one spacial dimension and the Ornstein-Uhlenbeck Process, a well known Itô-Diffusion process.
Finally, this poster presents numerical solutions to the stochastic heat equation in one and two spacial dimensions using both, finite elements and finite difference method.

The Problem and Definition of Solutions

Let U be an bounded domain in $\mathbb{R}^n$. We will consider a modification of the well known heat equation with homogeneous Dirichlet boundary conditions and given initial condition u_0 and random force term W.

$$\begin{cases} \partial_t u(t) - \Delta u = W(t,x,\omega), & \text{in } U \times (0,T) \\ u(0,x) = u_0(x), & \text{in } U \\ u(t) = 0, & \text{on } (0,T) \times \partial U. \end{cases} \tag{1}$$

Since the force term is a stochastic process, a solution to the stochastic heat equation will be a stochastic process as well.

Notation for Solutions of Stochastic Heat

The lack of regularity of many stochastic processes will require very weak notations of solutions. A predictable H-valued Process $\{u_t : t \in [0,T]\}$ with $u_t \in L^2(\Omega, H)$ is called a *strong solution* of the stochastic heat equation if

$$u_t = u_0 + \int_0^t \Delta u_s ds + \int_0^t \sigma dW_s, \quad \mathbf{P} - a.s., \tag{2}$$

a *weak solution* of the stochastic heat equation if for all $v \in \mathcal{D}(\Delta)$:

$$\langle u_t, v \rangle = \langle u_0, v \rangle + \int_0^t -\langle u_s, \Delta v \rangle ds + \int_0^t \langle \sigma dW_s, v \rangle, \quad \mathbf{P} - a.s., \tag{3}$$

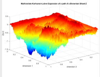
and a *mild solution* of the stochastic heat equation if

$$u_t = e^{-t\Delta} u_0 + \int_0^t e^{-(t-s)\Delta} \sigma dW_s, \quad \mathbf{P} - a.s.. \tag{4}$$

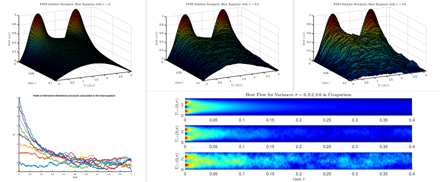
Where $e^{-t\Delta}$ denotes the semigroup generated by Δ.

The Random Force Term

To model the noise we will use a so called Q-Wiener Process. This is an $L^2(D)$-valued stochastic process. This figure shows a simulation of the process in a very illustrative way. Imagine dimension 2 is the time dimension. For each fixed time step you will obtain the graph of an element in $L^2(D)$. In this case, the operator Q is an integraloperator associated to the covariance function of Brownian motion.



Simulations in One Spacial Dimension using Finite Difference Method



Important Definitions and Theorems

Rather than seeing u as a function of time and space we want to see it as a $L^2(D)$-valued stochastic process. The random force term will defined as follows

Assumption on Q Let $Q \in \mathcal{L}(H,H)$ be non-negative definite, symmetric and such that there exists an orthonormal basis $\{\varphi_i : i \in \mathbb{N}\}$ of eigenfunctions with corresponding eigenvalues $\lambda_i \geq 0$ such that $\sum_{i\in\mathbb{N}} \lambda_i < \infty$.

Q-Wiener Process A H-valued stochastic process $\{W_t : t \geq 0\}$ is called Q-Wiener Process if

i) $W_0 = 0$ a.s.,
ii) W_t is a continious function $\mathbb{R}_+ \to H$ for each $\omega \in \Omega$,
iii) W_t is $\mathcal{F}_t$-adapted and $W_t - W_s$ is independent of $\mathcal{F}_s$ for $s \leq t$,
iv) $W_t - W_s \sim \mathcal{N}(0, (t-s)Q)$ for all $0 \leq s \leq t$.

Karhunen-Loève Expansion for Q-Wiener Process Let Q statisfy our basic assumptions. Then W_t is a Q-Wiener process if and only if

$$W_t = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \varphi_i \beta_t^{(i)} \quad a.s. \tag{5}$$

where $\beta^{(i)}$ are i.i.d. $\mathcal{F}_t$-Brownian motions and the series converges in $L^2(\Omega, H)$. Moreover it converges in $L^2(\Omega, \mathcal{C}([0,T], H))$.

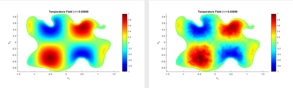
Stochastic Integral with respect to a Q-Wiener Process Using the Karhunen-Loève Expansion we define

$$\int_0^t X_s dW_s := \sum_{i=1}^{\infty} \sqrt{\lambda_i} \varphi_i \int_0^t X_s d\beta_s^{(i)}, \tag{6}$$

where the integrals with respect to Brownian motion are so called Itô integrals.

Simulations in Two Spacial Dimensions using Finite Element Method

The theoretical section to the stochastic heat equation in 2D shows, that there does not exist a weak solution when the force term is space-time white noise. However, since our numerical method interpolates in between discretization steps we can still obtain an interesting result. Unfortunately, we can not present a time-changing simulation of the numerical solution on a poster. Let us at least have a look on a fixed time step. Whereas the plot on the left is obtained by solving the equation with $\sigma = 0$, the on the right is result when we let the force term be random.



The Stochastic Heat Equation in 1D

We take $U = (0,\pi)$. Then $-\Delta$ has eigenfunctions and eigenvalues

$$\varphi_i(x) = \sqrt{2/\pi} \sin(ix), \qquad \lambda_i = i^2.$$

Let now W be a Q-Wiener process be such that Q has the same eigenfunctions as $-\Delta$ with corresponding eigenvalues ξ_i. Then for $v \in \mathcal{D}(\Delta)$ a weak solution statisfies:

$$\langle u_t, v \rangle_{L^2(U)} = \langle u_0, v \rangle_{L^2(U)} + \int_0^t \langle -u_s, \Delta v \rangle_{L^2(U)} ds + \sum_{i=1}^{\infty} \int_0^t \sigma \sqrt{\xi_i} \langle \varphi_i, v \rangle dB_s^{(i)}.$$

Expand $u_t = \sum_{i=1}^{\infty} \hat{u}_t^{(i)} \varphi_i$ for $\hat{u}_t^{(i)} := \langle u_t, \varphi_i \rangle_{L^2(U)}$ and take $v = \varphi_i$ to see

$$\hat{u}_t^{(i)} = \hat{u}_0^{(i)} - \int_0^t \lambda_i \hat{u}_s^{(i)} ds + \int_0^t \sigma \sqrt{\xi_i} dB_s^{(i)}. \tag{7}$$

Hence $\hat{u}^{(i)}$ is an Ornstein-Uhlenbeck Process. To simulate a weak solution to the stochastic heat equation we can thus simulate Ornstein Uhlenbeck Processes and compute the truncated sum in the above equation. One can show that $\mathrm{Var}(\hat{u}_t^{(i)}) = \frac{\sigma^2 \xi_i}{2\lambda_i}(1 - e^{-2\lambda_i t})$ and thus by Parseval's identity

$$\|u_t\|^2_{L^2(\Omega, L^2(0,\pi))} = \mathbb{E}\left[\sum_{i=1}^{\infty} |\hat{u}_t^{(i)}|^2\right] = \sum_{i=1}^{\infty} \frac{\sigma^2 \xi_i}{2\lambda_i}(1 - e^{-2\lambda_i t}) \tag{8}$$

which converges if $\sum_{i=1}^{\infty} \frac{\xi_i}{\lambda_i} < \infty$, which is the case since Q is trace class.

The Stochastic Heat Equation in 2D

Now let $U = (0,\pi) \times (0, pi)$. One can show that $-\Delta$ as the eigenvalues $\lambda_{i,j} = i^2 + j^2$. Again, let us assume that Q has the same eigenfunctions but with corresponding eigenvalues $\xi_{i,j}$. Expand and substitute to see

$$d\hat{u}^{(i,j)} = -\lambda_{i,j} \hat{u}^{(i,j)} dt + \sigma \sqrt{\xi_{i,j}} dB_t^{(i,j)}. \tag{9}$$

Once again, let us apply Parseval's identity to obtain

$$\|u_t\|^2_{L^2(\Omega, L^2((0,\pi)) \times (0,\pi))} = \mathbb{E}\left[\sum_{i,j=1}^{\infty} |\hat{u}_t^{(i,j)}|^2\right] = \sum_{i,j=1}^{\infty} \frac{\sigma^2 \xi_{i,j}}{2\lambda_{i,j}}(1 - e^{-2\lambda_{i,j} t}). \tag{10}$$

This justifies that u is in $L^2(\Omega, L^2((0,\pi) \times (0,\pi))$ since Q is trace class. However, for a cylindrical Wiener Process there is no solution since

$$\sum_{i,j=1}^{\infty} \frac{1}{\lambda_{i,j}} = \sum_{i,j=1}^{\infty} \frac{1}{i^2 + j^2} = \infty. \tag{11}$$

References

An Introduction to Computational Stochastic PDEs, J. Lord, E. Powell, T. Shardlow, August 2014, Camebridge University Press
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