ASSIGNMENT 7 Q3

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(1)
$e^{\sqrt{x}}$
$\frac{d}{dx}e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$
$e^{\sqrt{x}} = \sum_{n\geq 0} \frac{(\sqrt{x})^n}{n!} = \sum_{n\geq 0} \frac{x^{\frac{n}{2}}}{n!}$
$\frac{d}{dx}\sum_{n\geq 0} \frac{x^{\frac{n}{2}}}{n!} = \sum_{n\geq 1} \frac{x^{\frac{n-2}{2}}}{2(n-1)!} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

(2)
$cos(x^2)$
$\frac{d}{dx}cos(x^2) = -2xsin(x^2)$
$cos(x^2) = \sum_{n\geq 0} \frac{(-1)^n x^{4n}}{(2n)!}$
$\frac{d}{dx}\sum_{n\geq 0} \frac{(-1)^n x^{4n}}{(2n)!} = \sum_{n\geq 1} \frac{2(-1)^n x^{4n-1}}{(2n-1)!} = -2xsin(x^2)$

(3)
$\cos^2(x) + \sin^2(x) = 1$ is a trig identity.
we are going to prove this using series
we know that the maclaurin series for cos(x) is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$ and the maclaurin for sin(x) is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
The derivative of the maclaurin series of cos(x) is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$
The derivative of the maclaurin series of sin(x) is $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
when we differentiate $cos^2(x) + sin^2(x)$ we get $2sinxcosx - 2sinxcosx$ which is equal to 0.
so if we $\cos^2(x) + \sin^2(x) = k$. To find the value of the constant k we will use the series and substitute x with 0.
when we do the calculation $sin^2(0) + cos^2(0) = 1$

(4)

$$e^x \cdot e^y = e^{x+y}$$

Given the Maclaurin series,

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!},$$

$$e^x \cdot e^y = \left(\sum_{n\geq 0} \frac{x^n}{n!}\right) \cdot \left(\sum_{n\geq 0} \frac{y^n}{n!}\right) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \cdot \left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right)$$
$$= \left(1 + x + y + \frac{x^2}{2!} + \frac{y^2}{2!} + xy + \frac{x^3}{3!} + \frac{y^3}{3!} + \frac{x^2 y}{2!} + \frac{xy^2}{2!} \cdots\right)$$
$$= \left(1 + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \cdots\right) = \sum_{n\geq 0} \frac{(x+y)}{n!} = e^{x+y}$$

More rigorously...

$$\left(\sum_{n\geq 0}\frac{x^n}{n!}\right)\cdot\left(\sum_{n\geq 0}\frac{y^n}{n!}\right)=\sum_{n\geq 0}\sum_{k\geq 0}^{n}\frac{x^n}{n!}\cdot\frac{y^n-k}{(n-k)!}=\sum_{n\geq 0}\sum_{k\geq 0}^{n}\frac{1}{n!}\cdot\frac{n!}{k!(n-k)!}x^n y^{n-k}$$

$$=\sum_{n\geq 0}\frac{1}{n!}\sum_{k\geq 0}^{n}\frac{n!}{k!(n-k)!}x^n y^{n-k}$$

It follows

$$\sum_{n\geq 0}\frac{(x+y)^n}{n!}$$

by binomial theorem.