COMPLETING THE SQUARE

When given a quadratic function of the form $f(x) = ax^2 + bx + c$, we would like to write it in a form that tells us a bit more about the graph.

Recall: Vertex Form of a quadratic is when we have something of the form

$$\underbrace{a}_{\substack{a>1,\text{ stretch}\\0<a<1,\text{ compress}}}(x-\underbrace{h}_{\substack{\text{shift left, }h>0\\\text{right, }h<0}})^2 + \underbrace{k}_{\substack{\text{shift up, }k>0\\\text{down, }k<0}}$$

and the point (h,k) is the vertex.

So! Vertex form tells us a lot about the graph of $f(x)$; Really, all we need. If I move around the vertex form of $f(x)$, let's see what happens:

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a(x-h)(x-h) + k \\ &= a(x^2 - 2hx + h^2) + k \\ &= ax^2 - 2ahx + ah^2 + k \quad (*) \end{aligned}$$

(*) is of the form $ax^2 + bx + c$: Standard form! Then there must be a way to go from standard to vertex. This is done by completing the square!

Let's start out with an example:

$$f(x) = x^2 - 4x + 7$$

$f(x)$ is <u>not</u> a perfect square of the form $(x-\alpha)^2$

[[Cannot find ~~two~~ a number α so $\alpha \cdot \alpha = 7$ and $2\alpha = \alpha + \alpha = 4$]

But we can get it <u>close</u> to a perfect square!

Consider the first two terms:

$$f(x) = \underline{x^2 + 4x} + 7$$

We will play with this and get a perfect square!

① Take the ~~term~~ number in front of x, so 4, divide by 2, and square it.

$$\frac{4}{2} = 2 \quad \text{and} \quad 2^2 = 4$$

② Take the number you just found and add/subtract it to $f(x)$

note this is 0. nothing changed

$$f(x) = \underbrace{x^2 + 4x + 4}_{*} - 4 + 7$$

③ $*$ is a perfect square! $x^2 + 4x + 4 = (x+2)^2$

So,
$$f(x) = (x^2 + 4x + 4) - 4 + 7$$
$$= (x+2)^2 + 3 \longleftarrow \text{Vertex Form}$$

Lets make things a little harder:

$$f(x) = 2x^2 + 12x + 20$$

So, $a \neq 1$ (term in front of x^2). Let's keep the same process as in the last example in mind

① Work with the first two terms

$$f(x) = \underline{2x^2 + 12x} + 20$$

and to make life easier, factor out the 2 so:

$$f(x) = 2(x^2 + 6x) + 20$$

② Working in the parentheses, we want to make the inside a perfect square! So, as before, take the 6, divide by 2, and square it!

$$\frac{6}{2} = 3 \quad \text{and} \quad 3^2 = 9$$

So, $f(x) = \underline{2}(x^2 + 6x + 9) + 20 - \boxed{?}$

Rembeember before we added and subtracted the number we found. But now, if we add 9 and subtract 9, it won't be the same! Why? That 2 in the front makes a difference! I am really adding $2 \cdot 9 = 18$. So 18 is what I subtract! Then:

$$f(x) = 2(\underbrace{x^2 + 6x + 9}_{\text{Perfect square!}}) + 20 - 18 = 2(x+3)^2 + 2 \leftarrow \text{VERTEX FORM}$$

So using this technique, we can put any quadratic in standard form ax^2+bx+c, into vertex form $a(x-h)^2+k$ ((yes, that is the exact same a))

① Factor out a from the first two terms

$$f(x) = ax^2+bx+c = a\left(x^2+\frac{b}{a}x\right)+c$$

② Take the term in front of x, so $\frac{b}{a}$, divide it by 2, and square it

$$\left(\frac{b}{a}\right) \div 2 = \frac{b}{2a} \quad \text{and} \quad \left(\frac{b}{2a}\right)^2 = \boxed{\frac{b^2}{4a^2}}$$

③ Take your new number and ~~put~~ (add) it inside the parentheses and subtract $a \cdot \left(\frac{b^2}{4a^2}\right) = \frac{b^2}{4a}$ outside

$$f(x) = a\underbrace{\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)}_{}+c-\frac{b^2}{4a}$$

THIS WILL ALWAYS BE A PERFECT SQUARE

$$\frac{b}{2a}\cdot\frac{b}{2a} = \left(\frac{b}{2a}\right)^2 \quad \text{and} \quad \frac{b}{2a}+\frac{b}{2a} = \frac{2b}{2a} = \frac{b}{a}$$

④ Simplify!

$$f(x) = a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)+\underline{c-\frac{b^2}{4a}}$$

this is just some number

$$= a\left(x+\frac{b}{2a}\right)^2+k$$

You are now in vertex form!