Chain Rule Part 1, October 25, 2017

Warm-ups

1. Given $F(x) = f^2(g(x))$, $g(1) = 2, g'(1) = 3, f(2) = 4, f'(2) = 5$, find $F'(1)$.

$F'(x) = \frac{d}{dx}f^2(g(x)) = 2f(g(x)) \cdot f'(g(x)) \cdot g'(x)$

$F'(1) = 2f(g(1))f'(g(1)) \cdot g'(1) = 2f(2) \cdot f'(2) \cdot 3 = 2 \cdot 4 \cdot 5 \cdot 3 = 120$

2. Suppose $f(x)$ is differentiable such that $f(g(x)) = x$ and $f'(x) = 1 + (f(x))^2$. Find $g'(x)$.

$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(x) \rightsquigarrow f'(g(x)) \cdot g'(x) = 1$

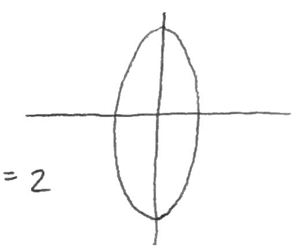
Since $f'(x) = 1 + (f(x))^2$, we have $f'(g(x)) \cdot g'(x) = (1 + (f(g(x)))^2)g'(x) = 1$

$\Rightarrow g'(x) = \frac{1}{\underbrace{1 + (f(g(x)))^2}_{\geq 1}} = \frac{1}{1+x^2}$

Problems:

1. Find the equation of the tangent lines to $4x^2 + y^2 = 72$ that are perpendicular to $2y + x + 3 = 0$.

$\frac{d}{dx}(4x^2+y^2) = \frac{d}{dx}(72) \rightsquigarrow 8x + 2yy' = 0 \rightsquigarrow y' = \frac{-4x}{y}$



The line we are considering is $y = \frac{-x}{2} - 3$, so need $y' = 2$.

If $y' = 2$, $2 = \frac{-4x}{y} \Rightarrow y = -2x$ } points on curve where $y' = 2$

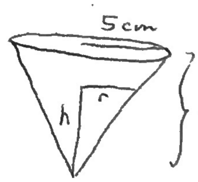
Substitute into curve: $4x^2 + 4x^2 = 72$

$x^2 = 9 \Rightarrow x = \pm 3$.

If $\left.\begin{array}{l} x = 3,\ y = 6 \\ x = -3,\ y = -6 \end{array}\right\}$ points of tangency $\rightsquigarrow$ $\begin{array}{l} y - 6 = 2(x-3) \\ y + 6 = 2(x+3) \end{array}$

2. James Bond orders a martini at a bar. How fast is the height of the liquid inside the glass changing?

- GLASS IS CONE 8cm deep, 10cm wide.
- Bartender pouring at a rate of $1\ \text{cm}^3/\text{sec} = V'(t)$.
- Time of interest is when glass is 3/4 full
- $V = \frac{1}{3}\pi r^2 h$



$\frac{r}{h} = \frac{5}{8}$

$r = \frac{5}{8}h$

$V_{TOTAL} = \frac{250}{3}\pi$, want $h'(t)$ when $V = \frac{250}{4}\pi$

$V(h) = \frac{1}{3}\pi\left(\frac{5}{8}h\right)^2 h = \frac{25}{192}\pi h^3$

$V'(t) = \frac{d}{dt}V(h(t)) = \frac{25}{192}\pi\, 3h(t)^2 h'(t) = 1\ \text{cm}^3/\text{min}$

When $V = \frac{250}{4}\pi$, $h^3 = 480 \Rightarrow h = \sqrt[3]{480}$

$h'(t) = \frac{1\ \text{cm}^3/\text{min} \cdot 192}{75\pi(\sqrt[3]{480})^2}$ (cm/min)