Series, September 22, 2017

Warm-ups

1. Does the series $\sum_{n\geq 1} \frac{\log(n)}{n}$ converge or diverge?

Note that $\log(n) > 1$ for $n \geq 3$. So, $0 \leq \frac{1}{n} < \frac{\log(n)}{n}$ for $n \geq 3$. Since $\sum_{n\geq 1} \frac{1}{n}$ diverges, the comparison test says $\sum_{n\geq 1} \frac{\log(n)}{n}$ must diverge as well.

2. Does the series $\sum_{n\geq 1} \left(\frac{\cos(n)}{n}\right)^2$ converge or diverge?

$0 \leq (\cos(n))^2 \leq 1$, so, $0 \leq \frac{(\cos(n))^2}{n^2} \leq \frac{1}{n^2}$. Since $\sum_{n\geq 1} \frac{1}{n^2}$ converges, so must $\sum \left(\frac{\cos(n)}{n}\right)^2$ by comparison.

3. Does the series $\sum_{n\geq 1} \frac{n}{\sqrt{n^4+7}}$ converge or diverge?

$0 \leq \frac{n}{\sqrt{n^4+7n^4}} = \frac{n}{n^2\sqrt{8}} = \frac{1}{n\sqrt{8}} \leq \frac{n}{\sqrt{n^4+7}}$. Since $\sum_{n\geq 1} \frac{1}{\sqrt{8}n}$ diverges, so must $\sum \frac{n}{\sqrt{n^4+7}}$ by comparison

More Problems

1. Does the series $\sum_{n\geq 1} \frac{\sqrt{n}}{2n^3+4}$ converge or diverge?

Take $a_n = \frac{1}{n^{5/2}}$. Then, $\lim_{n\to\infty} \frac{\sqrt{n}}{2n^3+4} \cdot \frac{n^{5/2}}{1} = \lim_{n\to\infty} \frac{n^3}{2n^3+4} = \frac{1}{2}$

Since a_n and $\frac{\sqrt{n}}{2n^3+4}$ are always positive, we can use the limit comparison test. The limit calculated is < 1, and since $\sum_{n\geq 1} \frac{1}{n^{5/2}}$ converges so must the series $\sum \frac{\sqrt{n}}{2n^3+4}$.

2. Does the series $\sum_{n\geq 1} \frac{4^n}{2^n+3^n}$ converge or diverge?

Take $a_n = \frac{4^n}{3^n}$ and $b_n = \frac{4^n}{2^n+3^n}$. $a_n, b_n > 0$, so we can use limit comparison. $\lim_{n\to\infty} \frac{4^n}{3^n} \cdot \frac{2^n+3^n}{4^n} = \lim_{n\to\infty} \frac{8^n+12^n}{12^n} = \infty$. So since $\sum \left(\frac{4}{3}\right)^n$ diverges as a geometric series with $r = 4/3 > 1$, so must $\sum \frac{4^n}{2^n+3^n}$.

3. For what values of p does the series $\sum_{n\geq 1} \frac{n^p}{n^3+2}$ converge?

Take $a_n = \frac{n^p}{n^3} = n^{p-3}$. $\sum_{n\geq 1} n^{p-3}$ converges if $p-3 < -1 \Rightarrow p < 2$.

$a_n, \frac{n^p}{n^3+2} > 0$, so we use the general comparison test.

$\frac{n^p}{n^3} > \frac{n^p}{n^3+2} \geq 0$. Since $\sum_{n\geq 1} \frac{n^p}{n^3}$ converges for $p<2$, so does $\sum_{n\geq 1} \frac{n^p}{n^3+2}$.