Continuity Part 2, October 6, 2017

PROBLEMS

1. Prove that there exists a vertical line that divides the following irregular shape in half.
   SOLUTION: Place this shape within the Cartesian plane, where x and y are both positive. Define $x = a$ as the left endpoint of the shape and $x = b$ as the right endpoint of the shape. Define $A(x)$ as the percentage of area to the left of the vertical line at x. We know that this function is continuous since the area of the shape is changing constantly, without any sudden jumps. Now, we note that $A(a) = 0$ since $x = a$ is the left endpoint. Also, we know that $A(b) = 1$ since $x = b$ is the right endpoint. Since $A(x)$ is continous, we can apply the *Intermediate Value Theorem* to it. Since $A(a) = 0$ and $A(b) = 1$, there must exist $c \in [a, b]$ such that $A(c) = 0.5$. Hence, there is a vertical line at $x = c$ that divides the area of the shape in half.

2. Prove that, given any circle, there exist two points opposite of each other that have the same temperature.
   SOLUTION: Place the cirlce in question in the Cartesian plane, such that the centre of the circle is placed at the origin. Let the function $T(P)$ be the temperature of the point P on the cirlce. Define $f(\theta) = T(P_1(\theta)) - T(P_2(\theta))$, where P_1 and P_2 are the points on the circle opposite of each other which intersect with the line that is an angle θ from the x-axis. Now, take some $\theta_1 \in [0, 2\pi]$. One of two things will happen. In the first case, we could have that $f(\theta_1) = 0$. If this is true, then $T(P_1(\theta_1)) - T(P_2(\theta_1)) = 0$ and we are done. Now, if $f(\theta_1) \neq 0$, then without loss of generality, $T(P_1(\theta_1)) - T(P_2(\theta_1)) > 0$. Then, consider $f(\theta_1 + \pi)$. We have that $f(\theta_1 + \pi) = T(P_2(\theta_1)) - T(P_1(\theta_1)) = -f(\theta_1)$. We know that $f(\theta)$ is continous, so we can apply the IVT. By the IVT, there must exist some θ_2 between θ_1 and $\theta_1 + \pi$ such that $f(\theta_2) = 0$. That is, there exist two points opposite of each other on the circle that have the same temperature.

3. Prove that if $f(x)$ is a continuous function on $[a, b]$ with $f(a), f(b) \in [a, b]$, then there exists $c \in [a, b]$ such that $f(c) = c$
   SOLUTION: Define $g(x) = f(x) - x$. We know that $g(x)$ is continous since $f(x)$ and $y = x$ are both continous functions and the difference of two continous functions must be continuous. Now, consider $g(a)$ and $g(b)$. We know that $g(a) = f(a) - a \geq 0$ since we must have $f(a) \in [a, b]$. Similarly, we have taht $g(b) = f(b) - b < 0$ since $f(b) \in [a, b]$. Since $g(x)$ is continous, we know that by the IVT, we must have $c \in [a.b]$ such that $g(c) = f(c) - c = 0$. This means we have found $c \in [a, b]$ such taht $f(c) = c$.