Continuity, October 4, 2017 SOLUTIONS

Definitions and Theorems

1. $f(x)$ is continuous at $x = a$ if ① $f(a)$ exists
   ② $\lim_{x \to a} f(x)$ exists
   ③ $\lim_{x \to a} f(x) = f(a)$
2. $(a, f(a))$ is a local max if $f(a) \geq f(x)$ for all x near a
3. $(a, f(a))$ is a global max if $f(a) \geq f(x)$ for all x
4. $(a, f(a))$ is a local min if $f(a) \leq f(x)$ for all x near a.
5. $(a, f(a))$ is a global min if $f(a) \leq f(x)$ for all x
6. THE EXTREME VALUE THEOREM: if $f(x)$ is continuous on $[l, r]$ then $f(x)$ attains a global extrema on $[l, r]$

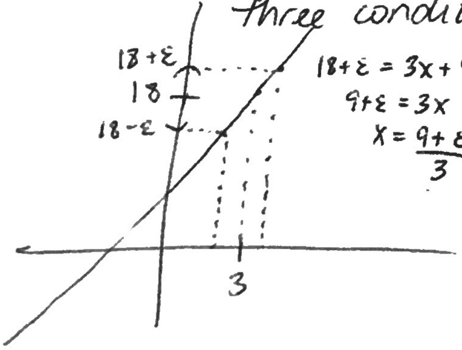
PROBLEMS

1. Mark the following students' work on this continuity problem out of 3 marks. Justify why you gave the mark you did.
   QUESTION: Prove that $f(x) = x^2$ is continous at $x = 2$. SOLUTION: If $f(x) = x^2$ is continous at $x = 2$, then it must satisfy three conditions. We see that $f(2) = 2^2$, so it exists and the first condition is satisfied. Secondly, we have that $\lim_{x \to 2} x^2 = 4$, so the second condition is satisfied. Finally, we have that $\lim_{x \to 2} x^2 = 4 = 2^2 = f(2)$, so the third condition is satisfied. Hence, $f(x) = x^2$ is continous at $x = 2$.

   * I would give it a 1/3. They assume what they want to prove.

2. Prove that $f(x) = 3x + 9$ is continous at $x = 3$

   If $f(x)$ is continuous at $x=3$, then we need to check all three conditions: ① $f(3) = 3(3) + 9 = 18$, so $f(3)$ exists



$18 + \varepsilon = 3x + 9$

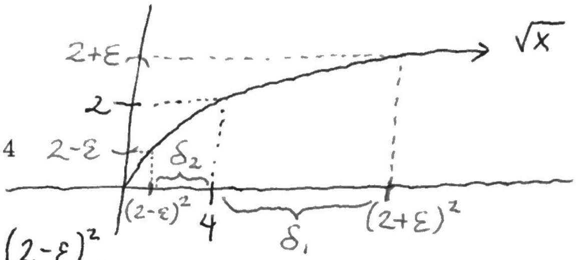
$9 + \varepsilon = 3x$

$x = \frac{9 + \varepsilon}{3} = 3 + \frac{\varepsilon}{3}$

② Claim $\lim_{x \to 3} f(x) = 18$

Let $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{3}$. Then if $x \in (3 - \frac{\varepsilon}{3}, 3 + \frac{\varepsilon}{3})$, $f(x) \in (18 - \varepsilon, 18 + \varepsilon)$ ▨

③ So $f(3) = 18 = \lim_{x \to 3} f(x)$.



Note $f(4)=\sqrt{4}=2$
So $f(4)$ is defined.

3. Prove that $f(x) = \sqrt{x}$ is continous at $x = 4$

Let $\varepsilon > 0$. Consider
$\delta_1 = (2+\varepsilon)^2 - 4$ and $\delta_2 = 4 - (2-\varepsilon)^2$.
For $0 < \varepsilon \leq 2$, we have $\delta_1 < \delta_2$. Take $\delta = \delta_2$. If $x \in (4-\delta, 4+\delta)$ then $f(x) \in (2-\varepsilon, 2+\varepsilon)$. For $\varepsilon > 2$, take $\delta = 2$. If $x \in (4-\delta, 4+\delta)$ then $f(x) \in (2-\varepsilon, 2+\varepsilon)$
Hence, $\lim_{x \to 4} \sqrt{x} = 2$ and $f(x) = \sqrt{x}$ is continuous at $x=4$.

4. Prove that $f(x) = \begin{cases} 0, & x \text{ is irrational} \\ 1, & x \text{ is rational} \end{cases}$ is discontinous at $x = \pi$

HINT: Consider $\{a_n\} = \{3, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \dots\}$

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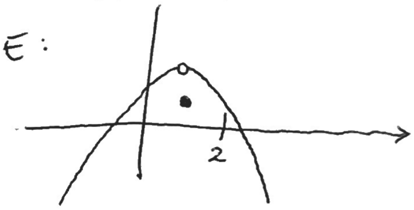
5. True or False: If $f(x)$ is defined everywhere, then $f(x)$ attains a global maximum on the interval $[0, 2]$.

FALSE:



this function is defined everywhere but has no global max on $[0,2]$. In fact, it has no global max on $(-\infty, \infty)$.

6. Find a function that satisfies all of the following conditions:
   (a) Discontinous at $x = 2$ and $x = 4$.
   (b) Local minimum at $x = \pi$
   (c) Global maximum at $x = 2$
   (d) No global minimum