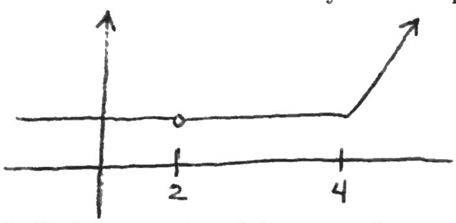
Definition of the Derivative, Recitation 10/13/2017

1. Sketch the graph, and come up with an expression for, a function that is continuous everywhere except $x = 2$ and differentiable everywhere except $x = 2$ and $x = 4$.

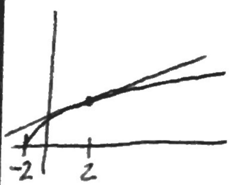


$$f(x) = \begin{cases} 1, & -\infty < x < 2 \\ 1, & 2 < x \leq 4 \\ x-3, & 4 > x \end{cases}$$

TAKEAWAY: Make your function simple!

2. Find the equation of the tangent line to $f(x) = \sqrt{x+2}$ at $x = 2$. Then, sketch the curve and the tangent line.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}. \quad \text{So, } f'(2) = \lim_{h\to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$$



$$f'(2) = \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \left(\frac{\sqrt{4+h} + \sqrt{4}}{\sqrt{4+h} + \sqrt{4}} \right) = \lim_{h \to 0} \frac{4+h-4}{(\sqrt{4+h}+2)h} = \lim_{h \to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$$

When $x=2$, $f(2) = 2$ and $f'(2) = \frac{1}{4}$. So the equation of the tangent line is $y - f(2) = f'(2)(x-2) \Rightarrow y = \frac{1}{4}x + \frac{3}{2}$

3. Let $f(x) = x^n$, where n is a positive integer. Using the definition of the derivative, find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \cdots + h^n - x^n}{h}$$

Polynomial in x and h

$$= \lim_{h \to 0} \frac{\cancel{h}(nx^{n-1} + hP(x,h))}{\cancel{h}}$$

$$= \lim_{h \to 0} nx^{n-1} + hP(x,h) = nx^{n-1}$$

$$\therefore \; f'(x) = nx^{n-1}$$

4. Challenge: Given $f(x) = \frac{1}{x-c}$, where c is a real number, prove that $f^{(n)}(x) = \frac{(-1)^n n!}{(x-c)^{n+1}}$ for all n.

BASE CASE $n=1$: $f'(x) = \lim_{h \to 0} \dfrac{\frac{1}{x+h-c} - \frac{1}{x-c}}{h} = \lim_{h \to 0} \dfrac{\frac{x-c-x-h+c}{(x+h-c)(x-c)}}{h}$

$$= \lim_{h \to 0} \frac{-1}{(x+h-c)(x-c)} = \frac{-1}{(x-c)^2} = \frac{(-1)^1 1!}{(x-c)^{1+1}} \checkmark$$

INDUCTIVE STEP: $n \Rightarrow n+1$ let $f^{(n)}(x) = \frac{(-1)^n n!}{(x-c)^{n+1}}$

$$f^{(n+1)}(x) = \lim_{h \to 0} \frac{f^{(n)}(x+h) - f^{(n)}(x)}{h} = \lim_{h \to 0} \frac{\frac{(-1)^n n!}{(x+h-c)^{n+1}} - \frac{(-1)^n n!}{(x-c)^{n+1}}}{h}$$

$$= \lim_{h \to 0} \frac{(-1)^{n+1} n!(n+1)}{(x+h-c)(x-c)^{n+1}} = \frac{(-1)^{n+1}(n+1)!}{(x-c)^{n+2}}$$