Sequences, September 13, 2017

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1. Prove, using the definition of convergence, that the following sequences converge

   (a) $\left\{\frac{1}{n}\right\}$

   SOLUTION: We wish to show that for every $\epsilon > 0$, there exists N such that if $n > N$, $\left|\frac{1}{n} - 0\right| < \epsilon$. Given $\epsilon > 0$, take $N > \frac{1}{\epsilon}$. Then, for $n > N$, $\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| < \left|\frac{1}{N}\right| < \epsilon$.

   (b) $\left\{\frac{n-1}{n+1}\right\}$

   SOLUTION: We wish to show that for every $\epsilon > 0$, there exists N such that if $n > N$, $\left|\frac{n-1}{n+1} - 1\right| < \epsilon$. Given $\epsilon > 0$, take $N > \frac{2}{\epsilon}$. Then, for $n > N$, $\left|\frac{n-1}{n+1} - 1\right| = \left|\frac{-2}{n+1}\right| < \left|\frac{2}{n}\right| < \left|\frac{2}{N}\right| < \epsilon$.

   (c) $\left\{\frac{1}{2^n}\right\}$

   SOLUTION: We wish to show that for every $\epsilon > 0$, there exists N such that if $n > N$, $\left|\frac{1}{2^n} - 0\right| < \epsilon$. Given $\epsilon > 0$, take $N > \frac{1}{\epsilon}$. Notice that for all $n > 0$, $2^n > n$. So, for $n > N$, $\left|\frac{1}{2^n} - 0\right| = \left|\frac{1}{2^n}\right| < \left|\frac{1}{n}\right| < \left|\frac{1}{N}\right| < \epsilon$.

   (d) $\left\{\frac{1}{6n^2+1}\right\}$

   SOLUTION: We wish to show that for every $\epsilon > 0$, there exists N such that if $n > N$, $\left|\frac{1}{6n^2+1} - 0\right| < \epsilon$. Given $\epsilon > 0$, take $N > \frac{1}{6\epsilon}$. Then, for $n > N$, $\left|\frac{1}{6n^2+1} - 0\right| < \left|\frac{1}{6n^2}\right| < \left|\frac{1}{6n}\right| < \left|\frac{1}{6N}\right| < \epsilon$.

   (e) $\left\{\frac{n}{n^2+1}\right\}$

   SOLUTION: We wish to show that for every $\epsilon > 0$, there exists N such that if $n > N$, $\left|\frac{n}{n^2+1} - 0\right| < \epsilon$. Given $\epsilon > 0$, take $N > \frac{1}{\epsilon}$. Then, for $n > N$, $\left|\frac{n}{n^2+1} - 1\right| < \left|\frac{n}{n^2}\right| = \left|\frac{1}{n}\right| < \left|\frac{1}{N}\right| < \epsilon$.

   (f) $\left\{\frac{3n+1}{2n+5}\right\}$

   SOLUTION: We wish to show that for every $\epsilon > 0$, there exists N such that if $n > N$, $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \epsilon$. Given $\epsilon > 0$, take $N > \frac{1}{2\epsilon}$. Then, for $n > N$, $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \left|\frac{3n+1}{2n} - \frac{3}{2}\right| = \left|\frac{1}{2n}\right| < \left|\frac{1}{2N}\right| < \epsilon$.

   (g) $\left\{\frac{1}{\sqrt{n}}\right\}$

   SOLUTION: We wish to show that for every $\epsilon > 0$, there exists N such that if $n > N$, $\left|\frac{1}{\sqrt{n}} - 0\right| < \epsilon$. Given $\epsilon > 0$, take $N > \frac{1}{\epsilon^2}$. Then, for $n > N$, $\left|\frac{1}{\sqrt{n}} - 0\right| < \left|\frac{1}{\sqrt{N}}\right| < \epsilon$.

2. Consider the sequence $\{a_n\}$, where $-A < a_n < A$ and $\{b_n\}$, where $\{b_n\}$ converges to 0. Does the sequence $\{a_n b_n\}$ converge?

   SOLUTION: The sequence $\{a_n b_n\}$ would converge and we suspect that it would converge to 0. You can draw some pictures to see this, or observe that the finite valued terms of a_n will eventually be multiplied by terms of b_n that are almost 0.

   By definition, we wish to show that for every $\epsilon > 0$, there exists N such that if $n > N$, $|a_n b_n - 0| < \epsilon$. We know that the sequence b_n converges to 0; that is, for every $\epsilon > 0$, there exists N such that if $n > N$, $|b_n - 0| < \epsilon$.

   Given $\epsilon > 0$, consider M such that for $n > M$, $|b_n| < \frac{\epsilon}{A}$ (we know this exists since b_n converges to 0). Then, for $n > M$, $|a_n b_n| < |A b_n| < A\,|b_n| < A\frac{\epsilon}{A} = \epsilon$