Sequences, Part 2, September 15, 2017

---

1. True or False: If $\{a_n\}$ is bounded, then it converges.

   SOLUTION: False. Take $(-1)^n$. This sequence is bounded between -1 and 1, but it does not converge (proven in recitation. Take $\epsilon = \frac{1}{2}$.)

2. True or False: If $\{a_n\}$ converges, then it is bounded.

   SOLUTION: This statement is true. Suppose the sequence a_n converges to L. This means, for every $\epsilon > 0$, there exists N such that if $n > N$, $|a_n - L| < \epsilon$. In particular, this is true for $\epsilon = 1$. For $\epsilon = 1$, we know there exists N such that if $n > N$, $|a_n - L| < 1$. That is, a_n is bounded between $L - 1$ and $L + 1$. So, the "tail" of the sequence is bounded. All we need to show is that the "head" of the sequence is bounded. The head of the sequence consists of the terms $a_1, a_2, \ldots, a_N$, which is a finite set. Take $U = \max\{a_1, a_2, \ldots, a_N, L+1\}$ and $B = \min\{a_1, a_2, \ldots, a_N, L-1\}$. We know that the tail is bounded above by $L+1$, so if $a_1, a_2, \ldots, a_N$ are all less than $L+1$, we take $U = L+1$ as the upper bound for the entire sequence. Conversely, if there are some $a_n > L+1$ where $1 \leq n \leq N$, take the maximum of those as the upper bound. The same argument follows for the lower bound. Hence, a_n is bounded.

3. Suppose $\{a_n\}$ diverges to $-\infty$ and $\{b_n\}$ converges to 1. Does $\{a_n b_n\}$ converge or diverge? Why?

   SOLUTION: Let's try to get a rough conceptual idea, first. When n gets really big, $a_n b_n$ will be something big and negative, multiplied by something close to 1. Thus, we suspect $a_n b_n$ to be something big and negative, thus diverging. Ok, so let's try and show that $a_n b_n$ diverges by definition.

   PROOF: Suppose $a_n b_n$ converges to L. That is, for every $\epsilon > 0$, there exists N such that if $n > N$, $|a_n b_n - L| < \epsilon$. Since b_n converges to 1, this implies that we can find N such that $b_n \neq 0$, for all $n > N$. Then, since $a_n b_n$ converges, the sequence with, for $n > N$, n^{th} term $\frac{1}{b_n} a_n b_n = a_n$ would also converge. However, this contradicts the fact that a_n diverges. Hence, the sequence $a_n b_n$ must diverge.

4. Suppose $\{a_n\}$ diverges to ∞ and $\{b_n\}$ converges to 0. Does $\{a_n b_n\}$ converge or diverge? Why?

   SOLUTION: This is an inconclusive statement. That is, the sequence $\{a_n b_n\}$ may or may not converge.Consider $a_n = n$ and $b_n = 1/n$. We know that a_n diverges to infinity and b_n converges to 0. Then, we have $a_n b_n = n \cdot 1/n = 1$, which is a convergent sequence. On the other hand, consider $a_n = n$ and $b_n = 1/\sqrt{n}$. We know that a_n diverges to infinity and b_n converges to 0. However, we have $a_n b_n = n \cdot \dfrac{1}{\sqrt{n}} = \sqrt{n}$, which diverges to infinity. See if you can come up with other examples!