The Mean Value Theorem, November 8, 2017

1. State Rolle's Theorem.

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Let $f(a)=f(b)$. Then, there exists $c \in (a,b)$ such that $f'(c)=0$.

2. State the Mean Value Theorem.

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then, there exists $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

3. Prove the Mean Value Theorem.

Suppose $f(x)$ satisfies the hypotheses of the MVT. That is, $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Define $g(x) = f(x) - L(x)$, where $L(x)$ is the line connecting $(a, f(a))$ & $(b, f(b))$. $L(x)$ has slope $\dfrac{f(b)-f(a)}{b-a}$. So, $g'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

Now, $g(x)$ satisfies the hypotheses of Rolle's theorem; it is continuous on $[a,b]$ and diff. on (a,b) (since f and L are). So, there exists $c \in (a,b)$ such that $g'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0 \iff f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Applications of the Mean-Value Theorem

4. Show that the function $f(x) = x^4 + 32x + k$, where k is a constant, has at most two real roots.

If a function has at most two roots, it cannot have 3. So, suppose $f(x)$ has three distinct roots, $x_1, x_2, x_3 \in \mathbb{R}$. That is, $x_1 \neq x_2 \neq x_3$ and $f(x_1) = f(x_2) = f(x_3) = 0$. $f(x)$ is continuous on $[x_1, x_3]$ and diff. on (x_1, x_3) so there must exist $r_1 \in (x_1, x_2)$ and $r_2 \in (x_2, x_3)$ s.t $f'(r_1) = f'(r_2) = 0$. But $f'(x) = 4x^3 + 32 = (x+2)(4x^2-8x+16)$ which only has $\underline{\text{ONE}}$ root. $\rightarrow\leftarrow$

5. Suppose $f(x)$ is differentiable and $f(4) = 5$. If for all x, $-3 \leq f'(x) \leq 2$, what are the possible values for $f(10)$?

By the mean value theorem, on $[4,10]$, there exists $c \in (4,10)$ such that $f'(c) = \dfrac{f(10)-f(4)}{6}$. Since $-3 \leq f'(x) \leq 2$ for all x, we have that

$$-3 \leq f'(c) \leq 2$$
$$-3 \leq \frac{f(10)-f(4)}{6} \leq 2$$
$$-18 \leq f(10) - f(4) \leq 12$$
$$-13 \leq f(10) \leq 17.$$

Consequences of the Mean Value Theorem

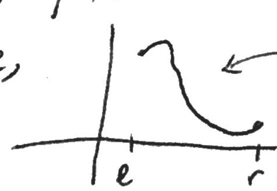
6. If $f'(x) < 0$ on an interval I, then $f(x)$ is decreasing on I.
7. If $f'(x) > 0$ on an interval I, then $f(x)$ is increasing on I.
8. If $f'(x) = 0$ on an interval I, then $f(x)$ is constant on I.

We will show that if $f'(x) < 0$ on I, then $f(x)$ is decreasing on I. The proofs of 7 and 8 are essentially identical.

6. Suppose $f'(x) < 0$ on I. Then, for any $a, b \in I = [l, r]$ we can apply the MVT and note that there exists $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a} < 0$. Since $b - a > 0$, we have $f(b) - f(a) < 0 \iff f(b) < f(a)$. Hence, $f(x)$ is decreasing on (a,b). Since we took a and b to be arbitrary, we see that $f(x)$ is decreasing on EVERY subinterval in I. Hence, $f(x)$ is decreasing on all of I.

! KEY POINT: just using the MVT on $[l, r]$ is not sufficent. Proving $f(r) < f(l)$ does not prove $f(b) < f(a)$ for any $a, b \in I$.

For Example,



this graph has $f(r) < f(l)$, but is not always decreasing.

To show decreasing on an interval, we need to show $f(x_1) < f(x_2)$ for ANY $x_2 < x_1$ that are contained in I.