Types of Optimization Problems, November 24, 2017

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1. A farmer has 180 ft of fencing and wants to build a rectangular pen using his barn as one side. What is the maximum area he can enclose?

   Solution: **Step 0:** Draw a picture. The situation above looks like this
   Step 1: Define the function you want to optimize. In this case, we have $Area = w \cdot l$.

   Step 2: Reduce your function to one variable, if necessary. Here, we see that we do have to reduce to one variable. Thankfully, we have that the perimeter P is fixed at 180 feet. So, $P = 180 = 2w + l$. Solving for l gives us that $l = 180 - 2w$, so $A = A(w) = w(180 - 2w) = 180w - 2w^2$, with $w \in (0, 90)$.

   Step 3: Differentiate and find critical points. Here we have that $A'(w) = 180 - 4w$. $A'(w)$ exists for all $w \in (0, \infty)$, so we just need to find where $A'(w) = 0$. This happens when $180 - 4w = 0$, so when $w = 45$.

   Step 4: Determine the min or max using either the *first derivative test* or the *Extreme Value Theorem*. In this problem, we note that our domain is $(0, \infty)$, so we need to use the $first derivative test$. For $w < 45$, $A'(w) > 0$ and for $w > 45$, we have that $A'(w) < 0$. Hence, we have that $w = 45$ yeilds a global maximum for $A(w)$.

   Step 5: Did you solve the problem? At this point, we haven't. The problem asks us to find the maximum area. All we need to do is plug in $w = 45$ into $A(w)$ to give us the maximum area of $A(45) = 180(45) - 2(45)^2 = 4050 ft^2$

2. You make a box out of a rectangle of cardboard by cutting four equal squares out of the corners and folding up the sides. Given a sheet 16" by 10", what is the maximum volume the box can have?
   Step 1: Define the function you want to optimize. If the length of the sides of the cut out squares is x, then the volume will be given by $V(x) = x(16 - 2x)(10 - 2x) = 4x^3 - 52x^2 + 160x$, where $x \in [0, 5]$.

   Step 2: Reduce your function to one variable, if necessary. Here, we note that we do not need to reduce to a single variable and can move on to step 3.

   Step 3: Differentiate and find critical points. We have a cubic function to optimize and have that $V'(x) = 12x^2 - 104x - 160$. This function exists for all x in $[0, 5]$, so we only need to determine where $V'(x) = 0$. This occurs when $x = 2$ or $x = 20/3$ (use the quadratic forumla to find these values). However, we note that $20/3 > 5$, so we can discard that critical point.

   Step 4: Determine the min or max using either the *first derivative test* or the *Extreme Value Theorem*. In this problem, we note that our domain is [0.5], so we want to use the *Extreme Value Theorem*. We must check the value of $V(x)$ at the endpoints 0 and 5, as well as at $x = 2$. We have $V(0) = 0$, $V(5) = 0$, and $V(2) = 144$. Hence, $V(2) = 144$ is the global max on $[0, 5]$

   Step 5: Did you solve the problem? The problem is asking us to determine the maximum volume, which is $144in^3$.

3. Find the minimum distance from the point $(3, 0)$ to the curve $y = x^2$
   Step 1: Define the function you want to optimize. We want to minimize the distance between a point and the function $y = x^2$. So, we want to minimize $D(x) = \sqrt{(x^2 - 0)^2 + (x - 3)^2} = \sqrt{x^4 + (x - 3)^2}$

Step 2: Reduce your function to one variable, if necessary. Here, we have one variable, but we can optimize D^2 instead of D, for ease of computation. We have that $d(x) = (D(x))^2 = x^4 + (x-3)^2$, where $x \in (-\infty, \infty)$.

Step 3: Differentiate and find critical points. We have that $d'(x) = 4x^3 + 2(x-3) = 2(x-1)(2x^2+2x+3)$. This function exits for all values of x, so we only need to determine where $d'(x) = 0$. This happens when $x = 1$

Step 4: Determine the min or max using either the *first derivative test* or the *Extreme Value Theorem*. We are on an open interval, so we will use the first derivative test. When $x < 1$, we have $d'(x) < 0$ and when $x > 1$ we have that $d'(x) > 0$. Hence, $x = 1$ yields a global minimum for $d(x)$.

Step 5: Did you solve the problem? At this point, no. We need to find them minimum distance, so we need to plug $x = 1$ into $D(x)$ (ask yourself why we aren't plugging into $d(x)$). We then have taht $D(1) = \sqrt{5}$

4. The sum of two nonnegative numbers is 36. Find the numbers such that the difference of their square roots is as large as possible.

   Step 1: Define the function you want to optimize. We want to maximize the difference of sqaure roots of two non-negative numbers, say a and b. So, we need to maximize $S = \sqrt{a} - \sqrt{b}$. We note that a and b must be in the interval $[0, 36]$

   Step 2: Reduce your function to one variable, if necessary. Here, we do have two variables in our function, so we need to reduce. Thankfully, we are given that $a + b = 36$. So, we have $a = 36 - b$. Subbing this into S yields $S(b) = \sqrt{36-b} - \sqrt{b}$, where $b \in [0, 36]$.

   Step 3: Differentiate and find critical points.
   We have that $S'(b) = \dfrac{-1}{2\sqrt{36-b}} - \dfrac{1}{2\sqrt{b}} = \dfrac{-\sqrt{b} - \sqrt{36-b}}{2\sqrt{b(36-b)}}$. This derivative exists for all $b \in (0, 36)$ and we note that $S'(b) \neq 0$ for any $b \in (0, 36)$.

   Step 4: Determine the min or max using either the *first derivative test* or the *Extreme Value Theorem*. We are on a closed interval $b \in [0, 36]$, so we will use the *Extreme Value Theorem*. Plugging in the endpoints $S(b)$ yields $S(0) = 6$ and $S(36) = -6$. We have that $b = 0$ or $b = 36$ yields a maximum difference (in absolute value).

   Step 5: Did you answer the question? At this point we have not. We were asked to find the pair of numbers a and b where $S = \sqrt{a} - \sqrt{b}$ was largest. This occurs when we have $a = 0, b = 36$ or $a = 36, b = 0$

5. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

   Step 1: Define the function you want to optimize. We want to optimize the volume of a cylinder. We know that $V_{cylinder} = \pi x^2 y$, where x is the radius of the cylinder and y is the height.

   Step 2: Reduce your function to one variable, if necessary. We have x and y as variables. Thankfully, we can notice that the height from the top of the cylinder to the top of the cone is $h - y$, so we can set up similar triangles and have that $\dfrac{h-y}{x} = \dfrac{h}{r}$, which yields $y = h - \dfrac{hx}{r}$. Hence, subbing this into V yields $V(x) = \pi x^2 (h - \dfrac{hx}{r}) = \pi h x^2 - \dfrac{\pi h x^3}{r}$, where $x \in [0, r]$.

Step 3: Differentiate and find critical points. We have that $V'(x) = 2\pi hx - \frac{3\pi hx^2}{r} = \pi hx(2 - \frac{3}{r}x)$. We note that $V'(x) = 0$ when $x = 0$ or when $x = \frac{2}{3}r$

Step 4: Determine the min or max using either the *first derivative test* or the *Extreme Value Theorem.* Since we are on a closed interval, we can use the Extreme Value Theorem and plug in our endpoints and critical points into $V(x)$. Then, we have $V(0) = 0$, $V(5) = 0$, and $V(\frac{2}{3}r) = \frac{4}{27}\pi r^2 h$. Hence, $V(\frac{2}{3}r)$ is a maximum.

Step 5: Did you solve the problem? We have that the maximum volume of a cylinder inscribed inside a cone with radius r and height h is $\frac{4}{27}\pi r^2 h$.